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Memoir on Abelian Transformations.

By Leonard Eugene Dickson.

Introduction.

One of the most important concepts relating to a group is the distribution of all of its operators into complete sets of conjugates within the group. It is presupposed, for example, in the recent theory of group-characters. For the general linear homogeneous group in an arbitrary field, this problem is readily treated* by means of the theory of canonical forms, as established by Jordan† for a field of prime order and by another method by the writer,‡ first for any finite field and later for an arbitrary field. By a supplementary discussion, the same method suffices for the group of linear homogeneous (or fractional) transformations of determinant unity, for the case of a finite field.|| For binary transformations, the case of an arbitrary field is treated in §1.

When we pass to a special class of groups, as the linear Abelian group, the problem is incomparably more difficult. The Abelian group affects an even number $2m$ of variables. The case $m = 1$ is treated in §1. The case $m = 2$ has been treated for a Galois field by the writer.§ The present paper aims to present a systematic general method of treatment for an arbitrary field, and to present in complete form the desired numerical results for $m = 3$ and the $GF[p^n]$. The paper presents a generalization to an arbitrary field of the canonical forms for $m = 2$, but does not duplicate the numerical work given in

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* Simple generalization of the papers by Dickson and Putnam on the ternary and quaternary groups, American Journal of Mathematics, Vol. XXIII, p. 37.

† Traité des Substitutions, pp. 114–126.

‡ American Journal, Vol. XXII, pp. 121–7; Vol. XXIV, pp. 101–8.

|| Dickson, American Journal, Vol. XXII, pp. 231–252; Putnam, ibid., Vol. XXIV.

§ Trans. Amer. Math. Soc., Vol. 2 (1901), pp. 103–138. Referred to as Tr. I had occasion to recheck all of the results of that paper.

the Transactions paper. The treatment in §§18–19 is much simpler than in the former paper. Moreover, §20, case (*i*), gives a necessary correction.* A few extra pages of this memoir therefore properly deal with this case $m=2$. Moreover, a large part of the work for $m=3$ consists of proof by induction from $m=1$ and $m=2$ to $m=3$.

In order to allow a more rapid and connected determination of the canonical types, I have reserved the determination of the number of non-conjugate forms of each type and the number of commutative transformations to a series of consecutive sections (§§37–42), which may conveniently serve for reference in place of the usual summarizing tables.

In certain cases I have found the completion of the discussion for an arbitrary field to be impracticable. I believe it highly desirable that an investigation should be made as far as expedient for an arbitrary domain, with explicit notice when specialization to a particular field (complex, real, rational, finite, ...) is made to secure needed properties.

As an incidental result, it may be noted that the senary Abelian group contains subgroups isomorphic with the general ternary linear group (§37), general ternary hyperorthogonal group (§40, type $T_{1\lambda}T_{2\lambda}T_{3\lambda}$), and the ternary orthogonal group (§39, type $L_{11}L_{21}L_{3\mu}$).

As to the importance of the linear Abelian group, one may mention the application to hyperelliptic and Abelian functions, and to various geometrical problems such as the 27 lines on a general cubic surface (group for $m=2$ $p^n=3$), the 28 bitangents to a quartic curve (group for $m=3$, $p^n=2$, a complete table for whose operators is given at the end of the paper).

If the modular theory is destined to be carried to a higher stage, the linear Abelian group in a finite field will play the same role as the congruence groups do in relation to the modular group.

I should state that the detailed steps of this paper have been carefully checked, while the considerably condensed character of §§37–41, which give the final results, led me to re-check this part. Very convincing checks are noted in §42.

1. Frequent use will be made of the canonical forms and conjugacies of

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* The statement (Tr., middle p. 129) that T replaces ϕ by ϕ'.

binary transformations of determinant unity in a general field F. Moreover, the group G of all such binary transformations is the group of binary special Abelian transformations (§2 for $m=1$).

Consider the transformation with coefficients in F

$$T:\quad x' = \alpha x + \gamma y,\quad y' = \beta x + \delta y, \qquad (\alpha\delta - \beta\gamma = 1).$$

Its characteristic equation is

$$\Delta(\rho) \equiv \rho^2 - d\rho + 1 = 0,\quad d \equiv \alpha + \delta.$$

Denote its roots by $\varkappa$, $\varkappa^{-1}$. Suppose first that $\varkappa$ belongs to F. Then according as $\varkappa \neq \varkappa^{-1}$ or $\varkappa = \varkappa^{-1}$, T is conjugate within G with the respective type

$$\begin{aligned} A:&\quad x' = \varkappa x,\quad y' = \varkappa^{-1} y;\\ B_\beta:&\quad x' = \pm x,\quad y' = \pm y + \beta x. \end{aligned}$$

Now $B_\beta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} B_\delta$ if, and only if,

$$b\beta = b\delta = 0,\quad d\beta = a\delta,\quad ad - bc = 1.$$

If $\delta \neq 0$, B_δ is not conjugate with B_0. If $\delta \neq 0$, and $\beta \neq 0$, then $b = 0$, $ad = 1$, $d\beta = a\delta$, so that B_β and B_δ are conjugate only if β/δ is the square of an element a of F. *Each* B_β *is conjugate within* G *with one of the non-conjugate types* B_0, B_1, B_{ν_i}, *where* ν_i *runs through a series of not-squares in* F *such that no two have as their ratio a square in* F, *while every not-square has with some* ν_i *a ratio which is a square. If* F *be the* $GF[p^n]$, *there are three types* B_0, B_1, B_ν *or two types* B_0, B_1, *according as* $p > 2$ *or* $p = 2$.

Suppose next that $\Delta(\rho)$ is irreducible in F, so that $\varkappa$ does not lie in F. Then $\gamma \neq 0$, and $y' = y + \alpha/\gamma$ transforms T into

$$S_\gamma:\quad x' = \gamma y,\quad y' = -\gamma^{-1} x + dy.$$

Introducing the new variables, conjugate with respect to F,

$$X = x - \gamma\varkappa y,\quad Y = x - \gamma\varkappa^{-1} y,$$

we obtain for S_γ the canonical form (not belonging to G)

$$C:\quad X' = \varkappa X,\quad Y' = \varkappa^{-1} Y.$$

Hence a transformation which transforms S_γ into S_{γ_1} merely multiplies X and Y by constants, and hence, if it belongs to G, must have the form

$$x' = ax - \gamma\gamma_1 by\,, \quad y' = bx + (\gamma\gamma_1^{-1}a - \gamma db)\,y\,,$$

where

$$\gamma\gamma_1^{-1}(a^2 - \gamma_1 dab + \gamma_1^2 b^2) = 1\,.$$

If $\gamma\gamma_1^{-1}$ is a square in F, the condition may be satisfied by taking $b = 0$. If $\gamma\gamma_1^{-1}$ is a not-square, then $b \neq 0$ and

$$k^2 - kd + 1 = \frac{1}{\gamma\gamma_1 b^2}\,, \quad k \equiv \frac{a}{b\gamma_1}\,.$$

For the $GF[p^n]$, $p > 2$, this condition can be satisfied. Indeed, $k^2 - kd + 1$, which is not zero in view of the irreducibility of $\Delta(\rho)$, has at least $\frac{1}{2}(p^n + 1)$ values, one of which is therefore a not-square in the field. For the corresponding value of k and a suitably chosen value of b, the condition will be satisfied. For the field of all real numbers, the condition can be satisfied if, and only if, $k^2 - kd + 1$ can be made negative, viz., if $d^2 > 4$.

The S_γ *are all conjugate within* F *if* F *is the* $GF[p^n]$, *or if* F *is the field* R *of all real numbers and* $d^2 > 4$. *For the field* R *and* $d^2 < 4$, *the* S *fall into two distinct sets of conjugates, represented by* S_1 *and* S_{-1}. *For a general* F, S_γ *and* S_{γ_1} *are conjugate if* γ/γ_1 *is a square in* F *or if* γ/γ_1 *is a not-square* ν *such that* $\nu(k^2 - kd + 1)$ *is a square for some value of* k *in the field* F, *but not conjugate in the remaining case.*

Definitions, Notations, Lemmas, §§2–5.

2. Consider two sets of variables ξ_i, x_i $(i = 1, \ldots, 2m)$ and the function

$$\psi_{\xi, x} = \sum_{l=1}^{m} \begin{vmatrix} \xi_{2l-1} & \xi_{2l} \\ x_{2l-1} & x_{2l} \end{vmatrix}\,. \tag{1}$$

Let the two sets of variables be transformed cogrediently as follows:

$$\xi_i' = \sum_{j=1}^{2m} \alpha_{ij}\xi_j\,, \quad x_i' = \sum_{j=1}^{2m} \alpha_{ij}x_j\,, \quad (i = 1, \ldots, 2m)\,. \tag{2}$$

Introducing the abbreviation, for r, s any distinct integers $\overline{<}\, 2m$,

$$C_{rs} = \sum_{l=1}^{m} \begin{vmatrix} \alpha_{2l-1\,r} & \alpha_{2l-1\,s} \\ \alpha_{2l\,r} & \alpha_{2l\,s} \end{vmatrix}\,, \tag{3}$$

involving the elements of the r^{th} and s^{th} columns of (α_{ij}), we find that

$$\psi_{\xi',x'}=\sum_{\substack{r,s=1,\ldots,2m\\ r<s}} C_{rs}\begin{vmatrix}\xi_r & \xi_s\\ x_r & x_s\end{vmatrix}. \tag{4}$$

The conditions for the identity $\psi_{\xi',x'}=\mu\psi_{\xi,x}$, μ being constant, are

$$C_{rs}=\begin{cases}\mu \ (\text{if } s=r+1 \quad\ =\text{even})\\ 0 \ (\text{unless } s=r+1=\text{even})\end{cases}(r,\, s=1,\ldots,2m;\ r<s). \tag{5}$$

The totality of transformations (α_{ij}), such that each α_{ij} and μ belong to a field F and satisfy the preceding conditions, constitutes the *general Abelian group* $GA(2m,\, F)$. Those of its transformations which have $\mu=1$, and hence leave $\psi_{\xi,x}$ absolutely invariant, form the *special Abelian group* $SA(2m,\, F)$.

The inverse of a general Abelian transformation $A=(\alpha_{ij})$ is

$$A^{-1}:\ \begin{bmatrix}\xi'_{2i-1}=\dfrac{1}{\mu}\displaystyle\sum_{j=1}^{m}(\alpha_{2j\,2i}\,\xi_{2j-1}-\alpha_{2j-1\,2i}\,\xi_{2j}),\\ \xi'_{2i}\quad\ =\dfrac{1}{\mu}\displaystyle\sum_{j=1}^{m}(-\alpha_{2j\,2i-1}\,\xi_{2j-1}+\alpha_{2j-1\,2i-1}\,\xi_{2j}).\end{bmatrix}(i=1,\ldots,m) \tag{6}$$

Indeed, the product AA^{-1} replaces ξ_{2i-1} by

$$\frac{1}{\mu}\sum_{k=1}^{m}(C_{2k-1\,2i}\,\xi_{2k-1}+C_{2k\,2i}\,\xi_{2k})\equiv\xi_{2i-1},$$

and likewise replaces ξ_{2i} by ξ_{2i}. Useful relations, together equivalent to the set (5), are obtained by determining directly the conditions that (6) shall make $\psi_{\xi',x'}=\dfrac{1}{\mu}\psi_{\xi,x}$. Evidently they may be obtained from (5) by replacing $\alpha_{2i-1\,2j-1}$ by $\dfrac{1}{\mu}\alpha_{2j\,2i}$, etc. The resulting relations (cf. Linear Groups, p. 116) are

$$R_{rs}=\begin{cases}\mu \ (\text{if } s=r+1 \quad\ =\text{even})\\ 0 \ (\text{unless } s=r+1=\text{even})\end{cases}(r,\, s=1,\ldots,2m;\ r<s) \tag{7}$$

where the symbol

$$R_{rs}=\sum_{l=1}^{m}\begin{vmatrix}\alpha_{r\,2l-1} & \alpha_{r\,2l}\\ \alpha_{s\,2l-1} & \alpha_{s\,2l}\end{vmatrix} \tag{8}$$

involves the elements of the r^{th} and s^{th} rows of the matrix (α_{ij}).

3. Lemma. *If, for any transformation* $A=(\alpha_{ij})$, $R_{2t-1\,2t}=\mu_t (t=1,\ldots\ldots,m)$ *and the remaining* R_{rs} *are all zero, then, for the inverse* A^{-1}, $C_{2t-1\,2t}=1/\mu_t$ $(t=1,\ldots\ldots,m)$ *and the remaining* C_{rs} *are all zero.*

It is first shown that the inverse A^{-1} is

$$\left.\begin{aligned} \xi'_{2i-1} &= \sum_{j=1}^{m} \frac{1}{\mu_j} (\alpha_{2j\,2i}\,\xi_{2j-1} - \alpha_{2j-1\,2i}\,\xi_{2j}), \\ \xi'_{2i} &= \sum_{j=1}^{m} \frac{1}{\mu_j} (-\alpha_{2j\,2i-1}\,\xi_{2j-1} + \alpha_{2j-1\,2i-1}\,\xi_{2j}). \end{aligned}\right\} (i=1,\ldots\ldots,m) \qquad (9)$$

In proof, we note that the product $A^{-1}A$ replaces ξ_k by

$$\sum_{j=1}^{m} \left(\frac{1}{\mu_{2j-1}} R_{k\,2j}\,\xi_{2j-1} + \frac{1}{\mu_{2j}} R_{2j-1\,k}\,\xi_{2j} \right) = \frac{1}{\mu_k} R_{2t-1\,2t}\,\xi_k = \xi_k,$$

upon setting $k=2t-1$ or $k=2t$, according as k is odd or even.

The values of C_{ij} for A^{-1} may now be derived from the values of the R_{ij} for A. For $j \overline{\gtrless} k$, we have from (9)

$$C_{2j-1\,2k} = \sum_{i=1}^{m} \frac{1}{\mu_j \mu_k} \begin{vmatrix} \alpha_{2j\,2i} & -\alpha_{2k-1\,2i} \\ -\alpha_{2j\,2i-1} & \alpha_{2k-1\,2i-1} \end{vmatrix} = \frac{-1}{\mu_j \mu_k} \sum_{i=1}^{m} \begin{vmatrix} \alpha_{2,\,2i-1} & \alpha_{2j\,2i} \\ \alpha_{2k-1\,2i-1} & \alpha_{2k-1\,2i} \end{vmatrix},$$

which equals $-\frac{1}{\mu_j \mu_k} R_{2j\,2k-1} = 0$ if $j<k$, but equals $+\frac{1}{\mu_j^2} R_{2j-1\,2j} = \frac{1}{\mu_j}$ if $j=k$. In what follows, let $j<k$. Then

$$C_{2j-1\,2k-1} = \frac{1}{\mu_j \mu_k} R_{2j\,2k} = 0, \qquad C_{2j\,2k-1} = \frac{-1}{\mu_j \mu_k} R_{2j-1\,2k} = 0,$$

$$C_{2j\,2k} = \frac{1}{\mu_j \mu_k} R_{2j-1\,2k-1} = 0.$$

Corollary. From (4) it follows that, for transformation $A=(2)$,

$$\psi_{\xi,x} = \sum_{t=1}^{m} \frac{1}{\mu_t} \begin{vmatrix} \xi'_{2t-1} & \xi'_{2t} \\ x'_{2t-1} & x'_{2t} \end{vmatrix}. \qquad (10)$$

4. Henceforth* we employ the standard notation

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* The notation (α_{ij}) was advantageous in §§2-3. Contrast (1) and (6) with (11) and (12) as to symmetry. One advantage in the notation (11) lies in the form of the inverse.

$$S:\quad \xi_i' = \sum_{j=1}^{m} (\alpha_{ij}\xi_j + \gamma_{ij}\eta_j), \quad \eta_i' = \sum_{j=1}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j) \qquad (i = 1, \ldots\ldots, m) \tag{11}$$

for a special Abelian transformation. In view of (6), its inverse is

$$S^{-1}:\quad \xi_i' = \sum_{j=1}^{m} (\delta_{ji}\xi_j - \gamma_{ji}\eta_j), \quad \eta_i' = \sum_{j=1}^{m} (-\beta_{ji}\xi_j + \alpha_{ji}\eta_j), \quad (i = 1, \ldots\ldots, m). \tag{12}$$

We shall usually designate S by its matrix. Thus for $m = 3$,

$$\begin{pmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} & \alpha_{13} & \gamma_{13} \\ \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \\ \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} & \alpha_{23} & \gamma_{23} \\ \beta_{21} & \delta_{21} & \beta_{22} & \delta_{22} & \beta_{23} & \delta_{23} \\ \alpha_{31} & \gamma_{31} & \alpha_{32} & \gamma_{32} & \alpha_{33} & \gamma_{33} \\ \beta_{31} & \delta_{31} & \beta_{32} & \delta_{32} & \beta_{33} & \delta_{33} \end{pmatrix}. \tag{13}$$

The symbols C_{rs} and R_{rs} being positional retain their meaning in the new notation. For a special Abelian transformation,

$$C_{2t-1\,2t} = R_{2t-1\,2t} = 1, \quad C_{rs} = R_{rs} = 0 \text{ (unless } s = r + 1 = \text{even).} \tag{14}$$

Formulae (76) and (78) of Linear Groups give the expanded form of (14).

Frequent use will be made of the following simple special Abelian transformations in the standard notation (unaltered variables being suppressed):

$$\begin{array}{lll} P_{ij} & = (\xi_i\xi_j)(\eta_i\eta_j); & M_i:\ \ \xi_i' = \eta_i, \quad \eta_i' = -\xi_i; \\ L_{i,\lambda}: & \xi_i' = \xi_i + \lambda\eta_i; & L_{i,\lambda}':\ \ \eta_i' = \eta_i + \lambda\xi_i; \\ N_{i,j,\lambda}: & \xi_i' = \xi_i + \lambda\eta_j, & \xi_j' = \xi_j + \lambda\eta_i; \\ Q_{i,j,\lambda}: & \xi_i' = \xi_i + \lambda\xi_j, & \eta_j' = \eta_j - \lambda\eta_i; \\ R_{i,j,\lambda}: & \eta_i' = \eta_i - \lambda\xi_j, & \eta_j' = \eta_j - \lambda\xi_i; \\ T_{i,\lambda}: & \xi_i' = \lambda\xi_i, & \eta_i' = \lambda^{-1}\eta_i. \end{array}$$

5. Lemma.* *In any field F, the characteristic equation*

$$\Delta(\rho) \equiv \begin{vmatrix} \alpha_{11} - \rho & \gamma_{11} & \alpha_{12} & \gamma_{12} \cdots \\ \beta_{11} & \delta_{11} - \rho & \beta_{12} & \delta_{12} \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \qquad (15)$$

of a special Abelian transformation S is a reciprocal equation.

From the form of the inverse (12) of S, we see that the first minors of α_{ij}, β_{ij}, γ_{ij}, δ_{ij} in $|S| \equiv \Delta(0)$ equal δ_{ij}, γ_{ij}, β_{ij}, α_{ij}, respectively. Since $\Delta(0) = 1$, it follows from the theory of determinants that any minor of order r whose diagonal elements all lie in the main diagonal of $|S|$ equals a similar minor of order $2m - r$. Hence if σ_r denotes the coefficient of $(-\rho)^{2m-r}$ in (15), $\sigma_r = \sigma_{2m-r}$.

Characteristic equation with a root in F, §§6–15.

6. Let S be a transformation of $G \equiv SA(2m, F)$ for which $\Delta(\rho) = 0$ has a root κ belonging to the given field F. Then there exists a linear homogeneous function ω with coefficients in F, not all zero, which S replaces by $\kappa\omega$. Moreover, G contains a transformation V which replaces ξ_1 by ω (Linear Groups, top of p. 93). Then $V^{-1}SV \equiv S_1$ replaces ξ_1 by $\kappa\xi_1$.

7. Suppose first that $\kappa \neq \kappa^{-1}$. There exists a function

$$\omega_1 = a_{11}\xi_1 + b_{11}\eta_1 + \cdots + a_{1m}\xi_m + b_{1m}\eta_m,$$

with coefficients in F and not reducing to $c\xi_1$, such that S_1 replaces ω_1 by $\kappa^{-1}\omega_1$.

If $b_{11} \neq 0$, G contains a transformation U_1 which replaces ξ_1 by $b_{11}^{-1}\xi_1$, and η_1 by ω_1 (Linear Groups, p. 93). Then $U_1^{-1}S_1U_1 \equiv S_2$ replaces ξ_1 by $\kappa\xi_1$ and η_1 by $\kappa^{-1}\eta_1$. In view of the Abelian conditions, it has the form

$$\xi_1' = \kappa\xi_1, \quad \eta_1' = \kappa^{-1}\eta_1, \quad \xi_i' = \sum_{j=2}^{m}(\alpha_{ij}\xi_j + \gamma_{ij}\eta_j), \quad \eta_i' = \sum_{j=2}^{m}(\beta_{ij}\xi_j + \delta_{ij}\eta_j), \qquad (16)$$
$$(i = 2, \ldots, m).$$

The problem is, therefore, reduced from m to $m - 1$ pairs of variables (§35, §37).

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* I first stated this theorem (with proof for $m = 2$) in the Transactions article cited in §1. A reviewer has suggested that it follows from the theory of bilinear alternating forms. But this theory does not seem to have been established for all fields, including those having a modulus. Note that for modulus 2, the expression of any form in terms of a symmetric and an alternating form is not valid.

If $b_{11}=0$, then a_{ij}, b_{ij} $(j=2, \ldots, m)$ are not all zero. According as a_{ij} or b_{ij} is not zero, we transform S_1 by P_{2j} or $P_{2j}M_2$ and obtain a similar transformation S_1' with $a_{12} \neq 0$. By the successive generality theorem,* G contains a transformation

$$\xi_1' = \xi_1, \quad \eta_1' = \eta_1 - a_{11}a_{12}^{-1}\eta_2, \quad \xi_2' = \omega_1, \ldots \tag{17}$$

This is seen to transform S_1' into

$$\left(\begin{array}{cccccccccc} \varkappa & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \beta_{11} & \varkappa^{-1} & \beta_{12} & 0 & \beta_{13} & \delta_{13} & \ldots & \beta_{1m} & \delta_{1m} \\ 0 & 0 & \varkappa^{-1} & 0 & 0 & 0 & \ldots & 0 & 0 \\ \beta_{21} & 0 & \beta_{22} & \varkappa & \beta_{23} & \delta_{23} & \ldots & \beta_{2m} & \delta_{2m} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right). \tag{18}$$

To obtain a similar transformation (18′) with $\beta_{11} = \beta_{22} = 0$, transform by

$$L_{1,\tau}' L_{2,\sigma}', \text{ where } \beta_{11} + \tau(\varkappa - \varkappa^{-1}) = 0, \quad \beta_{22} + \sigma(\varkappa^{-1} - \varkappa) = 0.$$

Let first $m=2$. If $\beta_{12}=0$, then $\beta_{21}=0$ and (18′) is a transformation (16). If $\beta_{12} \neq 0$, $T_{2,\sigma}$, where $\sigma = \beta_{12}\varkappa$, transforms (18′) into

$$\xi_1' = \varkappa\xi_1, \quad \eta_1' = \varkappa^{-1}\eta_1 + \varkappa^{-1}\xi_2, \quad \xi_2' = \varkappa^{-1}\xi_2, \quad \eta_2' = \varkappa\eta_2 + \varkappa\xi_1. \tag{19}$$

Let next $m \geqq 3$. Deleting the first two rows and columns of (18′), we obtain a special Abelian transformation A on ξ_i, η_i $(i=2, \ldots, m)$. Just as S_1 was conjugate with S_2 or (18′), so A is conjugate† with a transformation replacing ξ_2 by $\varkappa^{-1}\xi_2$, and η_2 by $\varkappa\eta_2$, or else with one replacing ξ_2 by $\varkappa^{-1}\xi_2$, and ξ_3 by $\varkappa\xi_3$. In the first case we transform by P_{12} and obtain B:

$$B = \left(\begin{array}{ccccccc} \varkappa^{-1} & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & \varkappa & \varkappa^2\beta_{12} & 0 & 0 & 0 & \ldots \\ 0 & 0 & \varkappa & 0 & 0 & 0 & \ldots \\ \beta_{12} & 0 & 0 & \varkappa^{-1} & \beta_{13} & \delta_{13} & \ldots \\ 0 & 0 & \alpha_{31} & 0 & \alpha_{33} & \gamma_{33} & \ldots \\ 0 & 0 & \beta_{31} & 0 & \beta_{33} & \delta_{33} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right).$$

In the second case, we transform by $L_{3,\rho}'$ and make $\beta_{33}=0$, obtaining C:

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* American Journal, Vol. XXIII, p. 365.

† By means of a special Abelian transformation on ξ_i, η_i $(i > 1)$.

$$C = \begin{pmatrix} \kappa & 0 & 0 & 0 & 0 & 0 \dots \\ 0 & \kappa^{-1} & \beta_{12} & 0 & \beta_{13} & \delta_{13} \dots \\ 0 & 0 & \kappa^{-1} & 0 & 0 & 0 \dots \\ \beta_{21} & 0 & 0 & \kappa & \beta_{23} & 0 \dots \\ \alpha_{31} & 0 & 0 & 0 & \kappa & 0 \dots \\ \beta_{31} & 0 & \beta_{32} & 0 & 0 & \kappa^{-1} \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

In B the first and second coefficients in the unwritten rows are zero by the Abelian conditions. If $\beta_{12} = 0$, B is of the form (16). If $\beta_{12} \neq 0$, we delete the first two rows and columns of B and obtain a special Abelian transformation B' on $\xi_i, \eta_i \ (i > 1)$. There exists a function

$$\omega_2 = a_{22}\xi_2 + b_{22}\eta_2 + \dots + a_{2m}\xi_m + b_{2m}\eta_m,$$

with coefficients in F and not reducing to $l\xi_2$, such that B' replaces ω_2 by $\kappa^{-1}\omega_2$. If $b_{22} \neq 0$, G contains a transformation U_2 which replaces $\xi_1, \eta_1, \xi_2, \eta_2$ by $\xi_1, \eta_1, b_{22}^{-1}\xi_2, \omega_2$, respectively. Then $U_2^{-1}BU_2$ is of the form B with $\beta_{1j} = \delta_{1j} = 0$ $(j = 3, \dots, m)$. Then $\alpha_{j1} = \beta_{j1} = 0$ $(j = 3, \dots, m)$ by the Abelian conditions. Transforming by $P_{12}\, T_{2,\sigma}$, where $\sigma = \beta_{12}\kappa$, we obtain

$$S_4 = \text{Product of (19) by a transformation on } \xi_i, \eta_i \ (i = 3, \dots, m). \tag{20}$$

The problem is therefore reduced from m to $m - 2$ pairs of variables (see §38). Next, if $b_{22} = 0$, we may take $a_{23} \neq 0$. Then G contains

$$\xi_1' = \xi_1, \ \eta_1' = \eta_1, \ \xi_2' = \xi_2, \ \eta_2' = \eta_2 - \alpha_{22}\alpha_{23}^{-1}\eta_3, \ \xi_3' = \omega_2, \dots \tag{17'}$$

This transforms B into a transformation replacing ξ_3 by $\kappa^{-1}\xi_3$, and otherwise of the form B. By Abelian conditions, $\delta_{33} = \kappa$, $\delta_{13} = 0$. Transforming by $L'_{3,\sigma}$, we make $\beta_{33} = 0$. Then transforming by $Q_{1,3,\lambda}$, where $\beta_{13} - \lambda\beta_{12} = 0$, we get

$$B_1 = \begin{pmatrix} \kappa^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \dots \\ 0 & \kappa & \kappa^2\beta_{12} & 0 & 0 & 0 & 0 & 0 \dots \\ 0 & 0 & \kappa & 0 & 0 & 0 & 0 & 0 \dots \\ \beta_{12} & 0 & 0 & \kappa^{-1} & 0 & 0 & \beta_{24} & \delta_{24} \dots \\ 0 & 0 & 0 & 0 & \kappa^{-1} & 0 & 0 & 0 \dots \\ 0 & 0 & \beta_{31} & 0 & 0 & \kappa & \beta_{34} & \delta_{34} \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{21}$$

the first, second, fourth and sixth coefficients in the unwritten rows being zero. If* $m = 3$, then $\beta_{31} = 0$ and $P_{13}^{-1}B_1P_{13}$ is of the form (16).

We next discuss C for the case $m = 3$. The Abelian conditions give

$$\beta_{31} = \beta_{13}, \quad \alpha_{31} = -\varkappa^2\delta_{13}, \quad \beta_{23} = \varkappa^2\beta_{32}, \quad \varkappa^{-1}\beta_{21} = \varkappa\beta_{12} - \delta_{13}\beta_{23}.$$

Now C is transformed into a similar transformation C' with $\beta_{31} = \beta_{13} = 0$ by

$$L'_{1,\sigma} R_{1,3,\tau}, \quad \tau(\varkappa - \varkappa^{-1}) = \beta_{13}, \quad \sigma(\varkappa - \varkappa^{-1}) = \tau(\alpha_{31} - \delta_{13}).$$

If $\beta_{32} = 0$, $P_{12}^{-1}C'P_{12}$ is of the form B. If $\beta_{32} \neq 0$, $Q_{3,1,\lambda}$, where $\beta_{12} - \lambda\beta_{32} = 0$, transforms C' into a similar transformation C'' with also $\beta_{12} = 0$. If $\delta_{13} = 0$, then C'' is of the form (16). If $\delta_{13} \neq 0$, C'' is transformed into

$$\begin{pmatrix} \varkappa & 0 & 0 & 0 & 0 & 0 \\ 0 & \varkappa^{-1} & 0 & 0 & 0 & \varkappa^{-1} \\ 0 & 0 & \varkappa^{-1} & 0 & 0 & 0 \\ -\varkappa & 0 & 0 & \varkappa & \varkappa & 0 \\ -\varkappa & 0 & 0 & 0 & \varkappa & 0 \\ 0 & 0 & \varkappa^{-1} & 0 & 0 & \varkappa^{-1} \end{pmatrix} \tag{22}$$

by $T_{2,\rho}\,T_{3,\sigma}$, where $\rho = \varkappa^2\delta_{13}\beta_{32}$, $\sigma = \varkappa^{-1}\delta_{13}^{-1}$.

We have now proved the preliminary

THEOREM.—*Within the group G, a transformation whose characteristic equation has a root $\varkappa \neq \varkappa^{-1}$ in the field F, is for $m = 2$, conjugate with a type* (16) *or* (19); *for $m = 3$, with a type* (16), (20) *or* (22).

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* If $m \overset{=}{>} 4$, we delete the first four rows and columns of B_1 and obtain a special Abelian transformation B_1' on ξ_i, η_i $(i = 3, \ldots, m)$. As above, B_1 is conjugate with a like transformation with $\beta_{3j} = \delta_{3j} = 0$ $(j = 3, \ldots, m)$, or else with one replacing ξ_3 by $\kappa^{-1}\xi_3$, and ξ_4 by $\kappa\xi_4$. In the first case, $\beta_{31} = 0$, so that its transform by P_{13} is of the form (16). In the second case, we transform by $L'_{4,\sigma}$ and make $\beta_{44} = 0$, the resulting transformation being

$$C_1 = \begin{pmatrix} \kappa^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \kappa & \kappa^2\beta_{12} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \kappa & 0 & 0 & 0 & 0 & 0 & \cdots \\ \beta_{12} & 0 & 0 & \kappa^{-1} & 0 & 0 & \beta_{14} & \delta_{14} & \cdots \\ 0 & 0 & 0 & 0 & \kappa^{-1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \beta_{32} & 0 & 0 & \kappa & \beta_{34} & \delta_{34} & \cdots \\ 0 & 0 & a_{42} & 0 & 0 & 0 & \kappa & 0 & \cdots \\ 0 & 0 & \beta_{42} & 0 & \beta_{43} & 0 & 0 & \kappa^{-1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

8. Let next $\Delta(\rho) = 0$ have a root $\kappa = \kappa^{-1}$, but no root μ in F such that $\mu \neq \mu^{-1}$. By the general theory of canonical forms, there exists a linear function ω_1 with coefficients in F and not reducing to $c\xi_1$, such that S_1 replaces ω_1 by $\pm\,\omega + \tau\xi_1$. If $b_{11} \neq 0$, G contains a transformation U_1 such that $U_1^{-1} S_1 U_1 \equiv L$ has the form (see §7)

$$\left.\begin{array}{l} \xi_1' = \pm\,\xi_1, \quad \xi_i' = \sum\limits_{j=2}^{m} (\alpha_{ij}\,\xi_j + \gamma_{ij}\,\eta_j), \\ \qquad\qquad \eta_1' = \pm\,\eta_1 + l\xi_1, \quad \eta_i' = \sum\limits_{j=2}^{m} (\beta_{ij}\,\xi_j + \delta_{ij}\,\eta_j), \end{array}\right\} (i = 2, \ldots, m), \qquad (23)$$

which is studied in §39. If $b_{11} = 0$, G contains a transformation (17) which transforms S_1 into R, where R replaces ξ_1 by $\pm\,\xi_1$, and ξ_2 by $\pm\,\xi_2 + \tau\xi_1$. Deleting the first two rows and columns in R, we obtain a special Abelian transformation R' on ξ_i, η_i $(i > 1)$, with $\xi_2' = \pm\,\xi_2$. Proceeding with R' as we did with S_1, we find that R is conjugate within G with one of the following two:

$$\left(\begin{array}{cccccc} \pm 1 & 0 & 0 & 0 & 0 & 0 \;\ldots \\ \beta_{11} & \pm 1 & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \;\ldots \\ \alpha_{21} & 0 & \pm 1 & 0 & 0 & 0 \;\ldots \\ \beta_{21} & 0 & \beta_{22} & \pm 1 & 0 & 0 \;\ldots \\ \alpha_{31} & 0 & 0 & 0 & \alpha_{33} & \gamma_{33} \;\ldots \\ \beta_{31} & 0 & 0 & 0 & \beta_{33} & \delta_{33} \;\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right), \qquad (24)_1$$

$$\left(\begin{array}{cccccc} \pm 1 & 0 & 0 & 0 & 0 & 0 \;\ldots \\ \beta_{11} & \pm 1 & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \;\ldots \\ \alpha_{21} & 0 & \pm 1 & 0 & 0 & 0 \;\ldots \\ \beta_{21} & 0 & \beta_{22} & \pm 1 & \beta_{23} & \delta_{23} \;\ldots \\ \alpha_{31} & 0 & \alpha_{32} & 0 & \pm 1 & 0 \;\ldots \\ \beta_{31} & 0 & \beta_{32} & 0 & \beta_{33} & \pm 1 \;\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right). \qquad (24)_2$$

9. Let first $m = 2$, so that $(24)_1$ and $(24)_2$ are identical. If $\beta_{12} = \delta_{12} = 0$, (24) is of the form (23). If $\delta_{12} = 0$, $\beta_{12} \neq 0$, the Abelian conditions give $\alpha_{21} = 0$, $\beta_{21} = \beta_{12}$. If also $\beta_{11} \neq 0$, the transform of (24) by $Q_{1,2,\kappa}$, where $\kappa = \beta_{12}\beta_{11}^{-1}$, is of the form (23). If $\beta_{11} = 0$, $\beta_{22} \neq 0$, we first transform (24) by P_{12} and obtain

a similar transformation with $\beta_{11} \neq 0$. If $\beta_{11} = 0$, $\beta_{22} = 0$, the transform of (24) by $T_{2,\,\beta_{12}}$ is M:

$$M = \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 1 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 1 & 0 & 0 & \pm 1 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 1 \\ -1 & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{pmatrix}.$$

If F does not have modulus 2, Z transforms M into

$$\xi_1' = \pm \xi_1, \quad \eta_1' = \pm \eta_1 + \xi_1, \quad \xi_2' = \pm \xi_2, \quad \eta_2' = \pm \eta_2 - \xi_2,$$

of the form (23). For modulus 2, $M = R_{1,2,1}$ furnishes a new type. Finally, if $\delta_{12} \neq 0$, the transform of (24) by $T_{2,\,\delta_{12}^{-1}}$ is of like form with $\delta_{12} = 1$. We then transform by $L'_{2,\,\beta_{12}}$ and obtain R_1:

$$R_1 = \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ \beta_{11} & \pm 1 & 0 & 1 \\ -1 & 0 & \pm 1 & 0 \\ \mp \alpha & 0 & \alpha & \pm 1 \end{pmatrix}, \quad A_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -\alpha & 0 & \alpha & 1 \end{pmatrix}. \qquad (25)$$

We assume first that F does not have modulus 2. Then $R_{1,2,\tau}$, where $\tau = -\frac{1}{2}\beta_{11}$, transforms R_1 into R_1', where R_1' is of the form R_1 with $\beta_{11} = 0$. For $\alpha = 0$, M_2 transforms R_1' into M. Let next $\alpha \neq 0$. For the upper signs, $R_1' = A_\alpha$. For the lower signs, $T_{1,-1}$ transforms R_1' into $A_{-\alpha}\, T_{1,-1}\, T_{2,-1}$. Now $T_{1,\kappa} T_{2,\kappa}$ transforms A_α into $A_{\alpha,\kappa^{-2}}$. Moreover, A_α and $A_{\alpha'}$ are not conjugate within $SA(4, F)$ if α'/α is a not-square. Indeed, $A_\alpha S = S A_{\alpha'}$ requires that

$$\gamma_{11} = \gamma_{12} = \gamma_{21} = \gamma_{22} = \alpha_{12} = \delta_{21} = 0, \quad \alpha_{22} = \alpha_{11}, \quad \delta_{22} = \delta_{11}, \quad \beta_{22} = \alpha\delta_{12},$$
$$\beta_{21} = -\beta_{12} - \alpha\delta_{12}, \quad \alpha\delta_{22} = \alpha'\alpha_{22}, \quad -\alpha'\alpha_{11} + \alpha'\alpha_{21} = -\beta_{22} - \alpha\delta_{22}.$$

Then Abelian condition R_{34} gives $\alpha_{22}\delta_{22} = 1$, whence $\alpha\delta_{22}^2 = \alpha'$. Moreover, A_α and $A_\beta T_{1,-1} T_{2,-1}$ are not conjugate since their characteristic equations have different roots.

Let next F be the $GF[2^n]$ or the infinite field $F^{(2)}$ defined as the aggregate of all the $GF[2^n]$, $n = 1, 2, 3, \ldots$. If $\alpha = 0$, the transform of R_1 by M_2 has the form (24) with $\delta_{12} = 0$, a case previously considered. If $\alpha \neq 0$, the transform of R_1 by $T_{1,\kappa} T_{2,\kappa}$ has the form R_1 with $\kappa^{-2}\alpha$ in place of α. Taking $\kappa = \alpha^{\frac{1}{2}}$, we obtain

$$R_\beta: \quad \xi_1' = \xi_1, \quad \eta_1' = \beta\xi_1 + \eta_1 + \eta_2, \quad \xi_2' = \xi_1 + \xi_2, \quad \eta_2' = \xi_1 + \xi_2 + \eta_2, \qquad (26)$$

of period 4. Now $L'_{2,\tau}\,Q_{2,1,\tau}$ transforms R_β into R_b where

$$b = \beta + \tau + \tau^2. \tag{27}$$

For the field $F^{(2)}$, (27) has a root τ in the field, so that the R_β are all conjugate. Let next F be the $GF[2^n]$. Then τ can be determined in F to make $b = 0$ if and only if β is a root of $f(\eta) = 0$, where

$$f(\eta) \equiv \eta^{2^{n-1}} + \eta^{2^{n-2}} + \ldots\ldots + \eta^4 + \eta^2 + \eta. \tag{28}$$

Indeed the condition is $f(\beta) = f(\tau) + f(\tau^2) = \tau^{2^n} + \tau = 0$ in the field. Hence the transformations R_β for which $f(\beta) = 0$ are all conjugate within $SA(4,\ 2^n)$. Likewise, the R_β for which $f(\beta) = 1$ are all conjugate. Lastly, no transformation R_β with $f(\beta) = 1$ is conjugate with R_0. For, if $R_\beta S = S R_0$, S must have the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \beta_{11} & 1 & \beta_{12} & \beta_{22} \\ \beta_{22} & 0 & 1 & 0 \\ \beta_{21} & 0 & \beta_{22} & 1 \end{pmatrix} \begin{pmatrix} \beta_{21} = \beta_{12} + \beta_{22} + \beta \\ \beta_{21} = \beta_{12} + \beta_{22}^2 \end{pmatrix}. \tag{29}$$

Hence must $\beta + \beta_{22} + \beta_{22}^2 = 0$ in the field, so that $f(\beta) = 0$.

THEOREM.—*For $m = 2$, (24) is conjugate with a type (23) or with*

R_0 or $R_{1,2,1}$, when F is the field $F^{(2)}$;
R_β or $R_{1,2,1}$, when F is the $GF[2^n]$,

where $\beta = 0$ or a particular root of $f(\eta) = 1$;

$$A_1,\ A_{v_i},\ A_1 T_{1,-1} T_{2,-1},\ A_v\, T_{1,-1} T_{2,-1},$$

when F does not have modulus 2, where v_i runs through a series of not-squares in F, such that no two have as their ratio a square, while every not-square has with some v_i a ratio which is a square.

10. Let next $m = 3$. Consider first $(24)_1$. Then

$$\begin{vmatrix} \alpha_{33} - \rho & \gamma_{33} \\ \beta_{33} & \delta_{33} - \rho \end{vmatrix} \equiv \rho^2 - \rho\,(\alpha_{33} + \delta_{33}) + 1 \tag{Q}$$

is a factor of $\Delta(\rho)$. By hypothesis, it has no root $\varkappa$ in the field F, such that $\varkappa \neq \varkappa^{-1}$. If $\varkappa = \pm 1$, we are led to $(24)_2$. If $\varkappa = \mp 1$, F not having modulus 2, we transform by P_{13} and find, as in §8, that $(24)_1$ is conjugate with a type (23). There remains the case in which (Q) is irreducible in F, so that $\gamma_{33} \neq 0$.

The transform of $(24)_1$ by $L'_{3,\lambda}$, where $\alpha_{33} - \lambda\gamma_{33} = 0$, is of the form $(24)_1$ with $\alpha_{33} = 0$:

$$\begin{pmatrix} \pm 1 & 0 & 0 & 0 & 0 & 0 \\ \beta_{11} & \pm 1 & \beta_{12} & \delta_{12} & \beta_{13} & \delta_1{}^3 \\ \alpha_{21} & 0 & \pm 1 & 0 & 0 & 0 \\ \beta_{21} & 0 & \beta_{22} & \pm 1 & 0 & 0 \\ \alpha_{31} & 0 & 0 & 0 & 0 & \gamma \\ \beta_{31} & 0 & 0 & 0 & -\gamma^{-1} & d \end{pmatrix}. \qquad (24)$$

Abelian condition R_{25} gives $\alpha_{31} = \pm\gamma\beta_{13}$. Transforming by $Q_{3,1,-\gamma\beta_{13}}$, we obtain a transformation $(24')$ with $\beta_{13} = 0$, $\alpha_{31} = 0$. We next show that we can take $\delta_{13} = 0$. If $\delta_{13} \neq 0$, we transform by $T_{1,\delta_{13}}$ and obtain a type $(24')$ with $\delta_{13} = 1$, and $\beta_{13} = \alpha_{31} = 0$ as before. Transforming the latter by

$$Q_{3,1,\pm\gamma a}R_{3,1,-a}, \qquad a\,(d \mp 2) = \pm\gamma^{-1},$$

we obtain a transformation differing from it only in the coefficients β_{11} and δ_{13}, and having $\delta_{13} = 0$. Note that a is determined in F since $d \mp 2 \neq 0$ in view of the irreducibility of (Q), which is now $\rho^2 - d\rho + 1$.

There results $(24')$ with $\beta_{13} = \delta_{13} = 0$. It is thus a product of the type treated in §9 by a transformation S_γ on ξ_3, η_3, treated in §1. Let F be the $GF[p^n]$. Then S_γ is conjugate with S_1 under transformation of determinant unity on ξ_3, η_3 within F. *Hence, for the* $GF[p^n]$, *the new canonical forms furnished by* $(24)_1$ *are* PS_1, *where* $P = R_\beta$ *or* $R_{1,2,1}$ *if* $p = 2$, $P = A_1$, A_ν, $A_1T_{1,-1}T_{2,-1}$ *or* $A_\nu T_{1,-1}T_{2,-1}$ *if* $p > 2$.

We may give to S_1 the ultimate canonical form $T_{3,\kappa}$, where $\kappa^{p^n+1} = 1$, κ belonging therefore to the $GF[p^{2n}]$.

Consider next the type $(24)_2$. Denote by R' the special Abelian transformation obtained from it by deleting the first two rows and columns. We may give to R' one of the canonical forms of §9 on ξ_2, η_2, ξ_3, η_3, upon applying a transformation of the latter variables. When R' is of the type (23), $(24)_2$ becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \beta_{11} & 1 & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \\ -\delta_{12} & 0 & 1 & 0 & 0 & 0 \\ \beta_{21} & 0 & \beta_{22} & 1 & 0 & 0 \\ -\delta_{13} & 0 & 0 & 0 & 1 & 0 \\ \beta_{31} & 0 & 0 & 0 & \beta_{33} & 1 \end{pmatrix}, \quad \begin{aligned} \beta_{21} &= \beta_{12} - \delta_{12}\beta_{22}, \\ \beta_{31} &= \beta_{13} - \delta_{13}\beta_{33}, \end{aligned} \qquad (30)$$

or its product by $T \equiv T_{1,-1} T_{2,-1} T_{3,-1}$, which changes the sign of each variable. Let first F have modulus 2. For $R' = R_\beta$ $(24)_2$ becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \beta_{11} & 1 & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \\ -\delta_{12} & 0 & 1 & 0 & 0 & 0 \\ \beta_{21} & 0 & \beta & 1 & 0 & 1 \\ \delta_{12}-\delta_{13} & 0 & -1 & 0 & 1 & 0 \\ \beta_{31} & 0 & -\alpha & 0 & \alpha & 1 \end{pmatrix}, \quad \begin{array}{l} \beta_{21} = \beta_{12} + \beta_{13} - \beta\delta_{12}, \\ \beta_{31} = \beta_{13} + \alpha\delta_{12} - \alpha\delta_{13}, \end{array} \qquad (31)$$

where $\alpha = 1$. For $R' = M_3^{-1} R_{2,3,1} M_3$, $(24)_2$ becomes (31) for $\beta = 0$, $\alpha = 0$. If F does not have modulus 2, we may take $R' = A_\alpha T_{1,\pm 1} T_{2,\pm 1}$, $\alpha = 1$ or ν_i. Then $(24)_2$ becomes (31) for $\beta = 0$, or its product by T.

11. Consider transformation (30). If $\delta_{12} \neq 0$, its transform by $R_{2,3,\lambda}$, where $\beta_{31} + \lambda\delta_{12} = 0$, differs from (30) only in β_{12}, β_{13}, β_{21}, β_{31}, with β_{31} now zero. When the result is transformed by $L'_{2,\sigma}$, $\beta_{21} - \sigma\delta_{12} = 0$, only β_{12} and β_{21} are altered, with $\beta_{21} = 0$. If also $\delta_{13} = 0$, we transform by P_{13} and obtain a transformation (23). Next, if $\delta_{12} = 0$, $\delta_{13} \neq 0$, we transform (30) by $R_{2,3,\lambda}$, where $\beta_{21} + \lambda\delta_{13} = 0$, and obtain a transformation (30) with $\delta_{12} = \beta_{21} = 0$, which leads to a (23). Hence there remain two cases:

$$\delta_{12} \neq 0, \ \delta_{13} \neq 0, \ \beta_{21} = 0, \ \beta_{31} = 0; \quad \delta_{12} = 0, \ \delta_{13} = 0, \ \beta_{21} \neq 0, \ \beta_{31} \neq 0.$$

If, in the second case, $\beta_{22} \neq 0$, we transform by $Q_{2,1,\lambda}$, $\beta_{12} - \lambda\beta_{22} = 0$, and obtain a similar transformation with $\beta_{12} = \beta_{21} = 0$. Likewise, if $\beta_{33} \neq 0$, we transform by $Q_{3,1,\lambda}$, $\beta_{13} - \lambda\beta_{33} = 0$, and have $\beta_{13} = \beta_{31} = 0$. But, if $\beta_{22} = \beta_{33} = 0$, it is transformed by $Q_{3,2,\lambda}$, $\beta_{21} - \lambda\beta_{31} = 0$, into a similar transformation with $\beta_{12} = \beta_{21} = 0$. Hence in each of these three sub-cases, we obtain a transformation (23). For the first main case, the Abelian conditions give

$$\beta_{12} = \delta_{12}\beta_{22}, \quad \beta_{13} = \delta_{13}\beta_{33}.$$

Transforming by $T_{2,\delta_{12}^{-1}} T_{3,\delta_{13}^{-1}}$, we obtain*

$$S_{b_1, b_2, b_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ b_1 & 1 & b_2 & 1 & b_3 & 1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & b_3 & 1 \end{pmatrix}. \qquad (32)$$

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* For properties of (32) see Amer. Math. Monthly, 1904, p. 88.

If $b_2 = b_3 = 0$, the transform of (32) by $Q_{3,2,-1}$ leaves ξ_3 and η_3 unaltered, and hence leads to (23). Since P_{23} transforms S_{b_1, b_2, b_3} into S_{b_1, b_3, b_2}, we may assume that $b_3 \neq 0$.

We seek the conditions under which (32) is conjugate within G with a transformation (23). In view of the successive generality theorem,* this conjugacy will follow if, and only if, new variables

$$X = a\xi_1 + b\eta_1 + c\xi_2 + d\eta_2 + e\xi_3 + j\eta_3,$$
$$Y = A\xi_1 + B\eta_1 + C\xi_2 + D\eta_2 + E\xi_3 + J\eta_3 \qquad (33)$$

can be determined such that

$$\begin{vmatrix} a & b \\ A & B \end{vmatrix} + \begin{vmatrix} c & d \\ C & D \end{vmatrix} + \begin{vmatrix} e & j \\ E & J \end{vmatrix} = 1, \qquad (34)$$

and such that, under transformation (32),

$$X' = X, \quad Y' = Y + tX, \qquad (35)$$

where $a, A, \ldots, J, t$ all belong to the field F. The conditions for (35) are

$$b = 0, \quad j = 0, \quad db_2 = 0, \quad c + e = 0,$$
$$B = 0, \quad J = te/b_3, \quad td = 0, \quad Db_2 = tc, \quad C + E = -ta.$$

If $b_2 \neq 0$, condition (17) becomes

$$tc^2(b_2^{-1} + b_3^{-1}) = 1,$$

whence $c \neq 0$, $b_2 + b_3 \neq 0$. Inversely, if $b_2 + b_3 \neq 0$, all the conditions can be satisfied. For example, we may take

$$X = \xi_2 - \xi_3, \quad Y = (b_3\eta_2 - b_2\eta_3)/(b_2 + b_3), \quad t = b_2b_3/(b_2 + b_3).$$

If $b_2 = 0$, the conditions reduce to

$$b = e = B = J = t = 0, \quad c + e = 0, \quad C + E = 0, \quad cD - dC = 1.$$

It suffices to take $c = D = 1$, $d = C = 0$, $a = A = 0$, whence $X = \xi_2 - \xi_3$, $Y = \eta_2$.

THEOREM.—*S_{b_1, b_2, b_3} is conjugate within G with (23) unless $b_3 = -b_2 \neq 0$.*

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* American Journal, Vol. XXIII, p. 365.

Consider next $S_{b_1, b_2, -b_2}$, where $b_2 \neq 0$. Now $T_{1, \kappa^{-1}} T_{2, \kappa^{-1}} T_{3, \kappa^{-1}}$ transforms S_{b_1, b_2, b_3} into $S_{\kappa^2 b_1, \kappa^2 b_2, \kappa^2 b_3}$. Hence if F be the $GF[2^n]$ or $F^{(2)}$, so that every element is a square, $S_{b_1, b_2, -b_2}$ is conjugate with a certain $S_{b, 1, 1}$. To obtain a like result when F does not have modulus 2, we note first that $R_{1, 2, \lambda}$ transforms S_{b_1, b_2, b_3} into $S_{b_1 + 2\lambda, b_2, b_3}$, so that we may give to b_1 an arbitrary value in F. Whether F has modulus 2 or not, the product

$$Q_{2, 3, -1} P_{12} N_{2, 3, b_1^{-1}} M_3 T_{2, -b_1 b_2^{-1}} T_{3, -b_2^{-1}} R_{2, 3, -\lambda} L'_{2, -2\lambda},$$

where $\lambda = b_2^2/b_1$, $b_1 \neq 0$, transforms $S_{b_1, b_2, -b_2}$ into $S_{b_2, \lambda, -\lambda}$. By choice of b_1, we may make $\lambda = 1$. Hence for any F, $S_{b_1, b_2, -b_2}$ is conjugate with $S_{b, 1, -1}$, where $b = 0$ if F does not have modulus 2.

If $S_{b, 1, -1} S = S S_{c, 1, -1}$, then S is found to have the form

$$\begin{pmatrix} \alpha_{11} & 0 & -\gamma & 0 & \gamma & 0 \\ \beta_{11} & \delta_{11} & \beta_{12} & \beta_{22} + \beta_{32} - c\gamma & \beta_{13} & -\beta_{23} - \beta_{33} - c\gamma \\ b\gamma - \beta_{22} - \beta_{23} & 0 & \alpha & \gamma & \alpha_{11} - \alpha & \gamma \\ \beta_{21} & \gamma & \beta_{22} & \alpha - \gamma & \beta_{23} & \alpha - \alpha_{11} - \gamma \\ b\gamma + \beta_{32} + \beta_{33} & 0 & \alpha - \delta_{11} & \gamma & \alpha_{11} + \delta_{11} - \alpha & \gamma \\ \beta_{31} & -\gamma & \beta_{32} & \delta_{11} - \alpha + \gamma & \beta_{33} & \alpha_{11} + \delta_{11} - \alpha + \gamma \end{pmatrix}, \quad (36)$$

$$\beta_{12} + \beta_{21} + \beta_{13} + \beta_{31} - \beta_{22} - \beta_{23} - \beta_{32} - \beta_{33} = b\delta_{11} - c\alpha_{11}. \quad (37)$$

The Abelian conditions are $R_{1j} \equiv 0 \; (j = 3, 4, 5, 6)$, $R_{35} \equiv 0$,

$$\begin{aligned} R_{12} \equiv {} & \alpha_{11}\delta_{11} - \gamma(\beta_{22} + \beta_{23} + \beta_{32} + \beta_{33}) = 1, \\ R_{23} \equiv {} & \gamma\beta_{12} + \gamma\beta_{13} - \alpha\beta_{22} - \alpha\beta_{32} + \delta_{11}\beta_{22} \\ & \qquad + \delta_{11}\beta_{23} + (\alpha_{11} - \alpha)(\beta_{23} + \beta_{33}) - b\gamma\delta_{11} + c\gamma\alpha_{11} = 0, \\ R_{24} \equiv {} & \gamma\beta_{11} - \delta_{11}\beta_{21} + (\alpha - \gamma)\beta_{12} + (\alpha - \alpha_{11} - \gamma)\beta_{13} \\ & \qquad - \beta_{22}(\beta_{22} + \beta_{32} - c\gamma) + \beta_{23}(\beta_{23} + \beta_{33} + c\gamma) = 0, \\ R_{34} \equiv {} & b\gamma^2 - \alpha_{11}\gamma - \alpha_{11}^2 + 2\alpha\alpha_{11} - 2\gamma\beta_{22} - 2\gamma\beta_{23} = 1, \\ R_{46} \equiv {} & -\gamma\beta_{21} - \gamma\beta_{31} + (\delta_{11} - \alpha + \gamma)\beta_{22} + (\gamma - \alpha)\beta_{32} \\ & \qquad + (\alpha_{11} + \delta_{11} - \alpha + \gamma)\beta_{23} + (\alpha_{11} - \alpha + \gamma)\beta_{33} = 0. \end{aligned}$$

Next, $R_{25} = 0$ reduces to $R_{12} = 0$, while $R_{26} + R_{24}$ gives

$$\begin{aligned} \delta_{11}(\beta_{12} + \beta_{13} - \beta_{21} - \beta_{31}) + c\gamma(\beta_{22} + \beta_{23} + \beta_{32} + \beta_{33}) & \\ - (\beta_{22} + \beta_{32})^2 + (\beta_{23} + \beta_{33})^2 = 0. & \quad (38) \end{aligned}$$

Further, $R_{36} + R_{34} = R_{12}$, $R_{56} + R_{34} = 2R_{12}$, $R_{23} - R_{46}$ equals γ times the left member of (37). Hence the conditions on (36) reduce to (37), (38) and those marked R_{12}, R_{24}, R_{34}, R_{46}.

Let F have modulus 2 and set $B \equiv \beta_{22} + \beta_{23} + \beta_{32} + \beta_{33}$. Then (37), R_{12}, R_{34} become

$$\beta_{12} + \beta_{21} + \beta_{13} + \beta_{31} = B + b\delta_{11} - c\alpha_{11}, \quad \alpha_{11}\delta_{11} - \gamma B = 1, \quad b\gamma^2 - \alpha_{11}\gamma - \alpha_{11}^2 = 1. \tag{39}$$

In view of these R_{46} and (38) become

$$\gamma(\beta_{21} + \beta_{31}) + B(\gamma + \alpha) + \delta_{11}(\beta_{22} + \beta_{23}) + \alpha_{11}(\beta_{23} + \beta_{33}) = 0, \quad B^2 + B\delta_{11} + b\delta_{11}^2 = c. \tag{40}$$

Hence, the conditions on (36) reduce to R_{24}, (39), (40). Let first $b \neq 0, c = 0$. Then $\delta_{11} \neq 0$ by $(39)_2$ and $(40)_2$, so that the latter may be written

$$(B/\delta_{11})^2 + B/\delta_{11} = b. \tag{41}$$

If F be the field $F^{(2)}$, (41) can always be solved in F. If F be the $GF[2^n]$, it can be solved in F if, and only if, $f(b) = 0$, f being defined by (28). Suppose this condition satisfied. Then, in either case, conditions R_{24}, (39) and (40), are all satisfied by

$$\alpha_{11} = \delta_{11} = 1, \quad \gamma = 0, \quad \alpha = 1, \quad \beta_{12} = \beta_{21} = \beta_{22} = \beta_{23} = \beta_{31} = \beta_{32} = 0, \quad \beta_{13} = B^2, \quad \beta_{33} = B. \tag{42}$$

Let next $f(b) = f(c) = 1$. Then $B^2 + B = b + c$ can be solved in F. For any such value of B and for the values (42), conditions R_{24}, (39), (40) are all satisfied.

THEOREM.—*If F does not have modulus 2, or if F is $F^{(2)}$, every $S_{b_1, b_2, -b_2}$ is conjugate with $S_{0,1,-1}$ within G. If F is the $GF[2^n]$, the $S_{b_1, b_2, -b_2}$ are conjugate with $S_{0,1,-1}$ or $S_{b,1,-1}$, b being a particular root of $f(\eta) = 1$, while the latter two are not conjugate.*

12. Consider next transformation (31). It is transformed by $Q_{2,1,\delta_{13}}$ into a similar transformation with $\delta_{13} = 0$. If $\alpha \neq 0$, the latter is transformed by $Q_{3,1,\lambda}$, $\beta_{13} - \lambda\alpha = 0$, into a similar transformation with $\beta_{13} = \delta_{13} = 0$. Now $R_{1,3,\lambda}$ transforms (31) into a transformation differing from it only in the coefficients β_{11}, β_{12}, β_{21}, having $\beta_{12} + \lambda$ in place of β_{12}, and $\beta_{21} + \lambda$ in place of β_{21}. Hence, if $\alpha \neq 0$, we may take $\beta_{13} = \delta_{13} = \beta_{12} = 0$; if $\alpha = 0$, we may take $\delta_{13} = \beta_{12} = 0$.

Consider, first, the case $\alpha \neq 0$, $\beta_{13} = \delta_{13} = \beta_{12} = 0$. If $\delta_{12} = 0$, (31) is of the form (23). If $\delta_{12} \neq 0$, $T_{1,\delta_{12}}$ transforms (31) into

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \beta' & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -\beta & 0 & \beta & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ \alpha & 0 & -\alpha & 0 & \alpha & 1 \end{pmatrix}. \tag{43}$$

To show that (43) is not conjugate within G with a transformation (23), we proceed as in §11. Let X and Y be defined by (33). The conditions for (35) are now

$$b = c = d = e = j = 0, \quad B = D = E = J = 0, \quad C = -ta,$$

so that (34) is impossible.

The subdivisions of the next sections are based on §10.

13. Consider (43) for $\beta = 0$, $\alpha \neq 0$, F not having modulus 2. We then denote (43) by $S_{\beta',\alpha}$. If S is the general transformation (13), $S_{\beta',\alpha} S = S S_{B,A}$ requires

$$\begin{aligned}
&\alpha_{12} = \alpha_{13} = \alpha_{23} = \delta_{21} = \delta_{31} = \delta_{32} = 0, \quad \delta_{11} = \delta_{22} = \delta_{33}, \quad \alpha_{11} = \alpha_{22} = \alpha_{33}, \\
&\alpha_{32} = \alpha_{21}, \quad \delta_{23} = \delta_{12}, \quad \alpha\delta_{11} = A\alpha_{11}, \quad \beta_{33} = \alpha\delta_{12}, \quad \gamma_{ij} = 0 \ (i, j, = 1, 2, 3), \\
&\beta_{23} = \alpha\delta_{13}, \quad \beta_{32} = -A\alpha_{31}, \quad \beta_{33} = -A\alpha_{21}, \\
&\beta_{22} = -\beta_{13} - \alpha\delta_{13}, \quad \beta_{32} = -\beta_{23} - \alpha\delta_{12}, \\
&\beta_{31} = \beta_{23} - \beta_{22} + \alpha\delta_{12}, \quad \beta'\delta_{11} - \beta_{12} + \beta_{13} + \alpha\delta_{13} = B\alpha_{11} + \beta_{21}.
\end{aligned}$$

Hence S must be of the form

$$\begin{pmatrix} \alpha_{11} & 0 & 0 & 0 & 0 & 0 \\ \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \\ \alpha_{21} & 0 & \alpha_{11} & 0 & 0 & 0 \\ \beta_{21} & 0 & -\beta_{13} - \alpha\delta_{13} & \delta_{11} & \alpha\delta_{13} & \delta_{12} \\ \alpha_{31} & 0 & \alpha_{21} & 0 & \alpha_{11} & 0 \\ \beta_{31} & 0 & -A\alpha_{31} & 0 & \alpha\delta_{12} & \delta_{11} \end{pmatrix}, \tag{44}$$

$$A\alpha_{11} = \alpha\delta_{11}, \quad A\alpha_{21} = -\alpha\delta_{12}, \quad A\alpha_{31} = \alpha\delta_{12} + \alpha\delta_{13}, \tag{45}$$

$$\beta_{31} = \beta_{13} + 2\alpha\delta_{13} + \alpha\delta_{12}, \quad \beta_{21} = \beta'\delta_{11} - B\alpha_{11} + \alpha\delta_{13} - \beta_{12} + \beta_{13}. \tag{46}$$

The conditions that (44) shall be Abelian then reduce to

$$\alpha_{11}\delta_{11} = 1, \quad \delta_{11}\delta_{12} + 2\delta_{11}\delta_{13} - \delta_{12}^2 = 0\,, \tag{47}$$

$$\alpha\delta_{12}\delta_{13} - \alpha\delta_{11}\delta_{13} - \alpha\delta_{13}^2 + 2\delta_{11}\beta_{12} + 2\delta_{12}\beta_{13} - \delta_{11}\beta_{13} - \beta'\delta_{11}^2 + B = 0. \tag{48}$$

Hence A/α must equal a square δ_{11}^2 in the field F. If this condition be satisfied, $S_{\beta',\alpha}$ is conjugate with $S_{B,A}$ within G. Indeed, we may employ (44) as transformer, where

$$\alpha_{11} = \delta_{11}^{-1}, \quad \beta_{11} = 0, \quad \delta_{12} = \delta_{13} = \alpha_{21} = \alpha_{31} = \beta_{12} = \beta_{21} = 0,$$
$$\beta_{31} = \beta_{13} = \alpha_{11}B - \delta_{11}\beta'.$$

Hence every $S_{\beta',\alpha}$ is conjugate within G with $S_{0,1}$ or one of the S_{0,ν_i}.

14. Consider (43) for $\alpha = 1$, F having modulus 2. Transforming it by $\xi_3' = \xi_3 - \eta_3$, then by $Q_{1,2,-1}$, and finally by $\eta_1' = \eta_1 + \beta'\xi_1$, we get

$$W_{\delta,\beta}\colon \quad \left\{\begin{matrix} \xi_1' = \xi_2, \quad \eta_1' = \delta\xi_1 + \eta_2, \quad \xi_2' = \xi_1, \\ \eta_2' = \beta\xi_1 + \eta_1 + \delta\xi_2 + \eta_3, \quad \xi_3' = \eta_3, \quad \eta_3' = \xi_1 + \xi_3. \end{matrix}\right\} \tag{49}$$

If $W_{\delta,\beta}S = SW_{D,B}$, S must have the form

$$\begin{pmatrix} \alpha_{11} & 0 & \alpha_{12} & 0 & 0 & 0 \\ \beta_{11} & \alpha_{11} & \beta_{12} & \alpha_{12} & \beta_{13} & \delta_{13} \\ \alpha_{12} & 0 & \alpha_{11} & 0 & 0 & 0 \\ \beta_{21} & \alpha_{12} & \beta_{22} & \alpha_{11} & \delta_{13} & \alpha_{12} + \beta_{13} \\ \alpha_{31} & 0 & \alpha_{12} + \beta_{31} & 0 & \alpha_{11} & \alpha_{12} \\ \beta_{31} & 0 & \alpha_{31} & 0 & \alpha_{12} & \alpha_{11} \end{pmatrix},$$

$$\left.\begin{aligned} \beta_{22} &= \beta_{11} + D\alpha_{12} + \delta\alpha_{12}, \\ \beta_{21} &= \delta_{13} + \beta_{12} + \beta\alpha_{12} + D\alpha_{11} + \delta\alpha_{11}, \\ \beta_{31} &= \beta_{13} + \alpha_{12} + B\alpha_{11} + \beta\alpha_{11}, \\ \alpha_{31} &= \delta_{13} + B\alpha_{12} + \beta\alpha_{12}, \end{aligned}\right\} \tag{50}$$

The Abelian conditions on (50) reduce to

$$\alpha_{11} + \alpha_{12} = 1, \quad \alpha_{11}^2 + \alpha_{11} + B + \beta = 0\,, \tag{51}$$

$$\delta_{13}^2 + \alpha_{11}\delta_{13} + \beta_{13}^2 + \alpha_{12}\beta_{13} + D + \delta + \alpha_{11}\alpha_{12}\beta = 0\,. \tag{52}$$

If F be the field $F^{(2)}$, $(51)_2$ can be solved for α_{11} in F. If F be the $GF[2^n]$, the condition for solution is $f(B)=f(\beta)$. Suppose this condition satisfied. In either case, (52) can be solved in F. For example, we may take*

$$\beta_{13}=\delta_{13}=D+\delta+\alpha_{11}\alpha_{12}\beta. \tag{53}$$

THEOREM.—*For the field $F^{(2)}$, every $W_{\delta,\beta}$ is conjugate with $W_{0,0}$. For the $GF[2^n]$, $W_{\delta,\beta}$ is conjugate with $W_{0,0}$ or $W_{0,b}$, where b is a particular root of $f(\eta)=1$, the two being not conjugate within G.*

15. It remains to consider (31) for $\alpha=\beta=0$, F having modulus 2. By §12, we may take $\beta_{12}=\delta_{13}=0$. Then (31) may be written

$$V_{\gamma,\delta,\epsilon}:\ \left\{\begin{array}{lll}\xi_1'=\xi_1, & \eta_1'=\gamma\xi_1+\eta_1+\delta\eta_2+\epsilon\xi_3, & \xi_2'=\delta\xi_1+\xi_2\\ \eta_2'=\epsilon\xi_1+\eta_2+\eta_3, & \xi_3'=\delta\xi_1+\xi_2+\xi_3, & \eta_3'=\epsilon\xi_1+\eta_3.\end{array}\right\} \tag{54}$$

To test its conjugacy with (23), we proceed as in §11. If $\delta=\epsilon=0$, V is of the form (23). Let next δ and ϵ not both $=0$. Then (35) requires

$$b=d=e=0,\quad c\delta+j\epsilon=0,\quad B=0,$$
$$E=tc,\quad D=tj,\quad C\delta+D\epsilon+E\delta+J\epsilon=ta.$$

Then (34) becomes $1=cD-jE=0$ and is impossible, so that the two are not conjugate within the Abelian group.

Since $P_{23}M_2M_3$ transforms $V_{\gamma,\delta,\epsilon}$ into $V_{\gamma,\epsilon,\delta}$, we may assume that $\delta\neq 0$. Then the Abelian transformation

$$\xi_1'=\xi_1,\quad \eta_1'=\eta_1+\gamma/\delta\xi_3,\quad \xi_2'=1/\delta\xi_2,$$
$$\eta_2'=\gamma/\delta\xi_2+\delta\eta_2+\epsilon\xi_3,\quad \xi_3'=1/\delta\xi_3,\quad \eta_3'=\gamma\xi_1+\epsilon\xi_2+\delta\eta_3,$$

transforms $V_{\gamma,\delta,\epsilon}$ into $V_{0,1,0}$. Further, the Abelian transformation

$$\xi_1'=\xi_1+\eta_3,\quad \eta_1'=\xi_3,\quad \xi_2'=\xi_1+\xi_2+\eta_2,$$
$$\eta_2'=\eta_1+\xi_3,\quad \xi_3'=\xi_2+\eta_2,\quad \eta_3'=\eta_1+\xi_2+\xi_3$$

transforms $V_{0,1,0}$ into $S_{0,1,1}$.

Characteristic equation with no root in F, §§16–33.

16. THEOREM.—*Within $G\equiv SA(2m, F)$, any transformation S whose characteristic equation has no root in F, is conjugate with a transformation which replaces ξ_1 by $\gamma\eta_1$.*

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* As a check, I verified that, when (51) and (53) hold and $\beta_{11}=\beta_{12}=0$, transformation (50) is Abelian and transforms $W_{\delta,\beta}$ into $W_{D,B}$.

Let S replace ξ_1 by $\omega \equiv \alpha_{11}\xi_1 + \gamma_{11}\eta_1 + \cdots$. If $\gamma_{11} \neq 0$, there exists in G a transformation V which replaces ξ_1 by $\gamma_{11}^{-1}\xi_1$ and η_1 by ω. Then $V^{-1}SV$ replaces ξ_1 by $\gamma_{11}^{-1}\eta_1$. The same result follows if S is conjugate within G with a transformation having $\gamma_{11} \neq 0$. Suppose then that $\gamma_{11} = 0$ in all the conjugates to S. The transform of S by M_1, P_{1i} or $P_{1i}M_1$, where $i > 1$, has β_{11}, γ_{ii} or β_{ii} respectively, as the coefficient of η_1 in ξ_1'. Hence, every $\beta_{ii} = \gamma_{ii} = 0$ $(i = 1, \ldots, m)$. The transform of the resulting transformation S' by $\eta_i' = \eta_i + \eta_j$, $\xi_j' = \xi_j - \xi_i$ has $\beta_{ij} + \beta_{ji}$ as the coefficient of ξ_i in η_i', and $-\gamma_{ij} - \gamma_{ji}$ as the coefficient of η_j in ξ_j'. Hence

$$\beta_{ij} = -\beta_{ji}, \quad \gamma_{ij} = -\gamma_{ji}, \qquad (i, j = 1, \ldots, m;\ i \neq j).$$

The transform of S' by $\xi_i' = \xi_i + \eta_j$, $\xi_j' = \xi_j + \eta_i$ has $\delta_{ji} - \alpha_{ij}$ as the coefficient of η_i in ξ_i'. Hence $\alpha_{ij} = \delta_{ji}$ for $i \neq j$. Finally, the transform of S' by $L_{i,1}$ has $\delta_{ii} - \alpha_{ii}$ as the coefficient of η_i and ξ_i'. Hence $\alpha_{ii} = \delta_{ii}$. It follows from §4 that S' is of period 2. By the general theory of canonical forms, the roots of the characteristic equation of S' all satisfy $\kappa^2 = 1$, contrary to hypothesis.

17. Let S be a transformation of G which replaces ξ_1 by $\gamma\eta_1$, and for which $\Delta(\rho) = 0$ has no root in F. Thus S replaces η_1 by

$$-\gamma^{-1}\xi_1 + \delta\eta_1 + \tau, \quad \tau \equiv \beta_{12}\xi_2 + \delta_{12}\eta_2 + \cdots + \beta_{1m}\xi_m + \delta_{1m}\eta_m.$$

If $\tau \equiv 0$, S becomes, in view of the Abelian conditions

$$\left.\begin{aligned} \xi_1' = \gamma\eta_1, \quad \xi_i' &= \sum_{j=2}^{m} (\alpha_{ij}\xi_j + \gamma_{ij}\eta_j), \\ \eta_1' = -\gamma^{-1}\xi + \delta\eta_1, \quad \eta_i' &= \sum_{j=2}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j), \end{aligned}\right\} \quad (i = 2, \ldots, m,) \qquad (55)$$

so that the problem is reduced from m to $m - 1$ pairs of variables (see §40). If $\tau \neq 0$, there exists a transformation Σ on ξ_i, $\eta_i (i > 1)$, which replaces ξ_2 by τ. Then $\Sigma^{-1}S\Sigma = S_1$ replaces ξ_1 by $\gamma\eta_1$ and η_1 by $-\gamma^{-1}\xi_1 + \delta\eta_1 + \xi_2$.

18. For $m = 2$, S_1 becomes, in view of the Abelian conditions,

$$S_1 = \begin{pmatrix} 0 & \gamma & 0 & 0 \\ -\gamma^{-1} & \delta & 1 & 0 \\ 0 & \gamma\gamma_{22} & \alpha_{22} & \gamma_{22} \\ 0 & \gamma\delta_{22} & \beta_{22} & \delta_{22} \end{pmatrix}, \quad \alpha_{22}\delta_{22} - \beta_{22}\gamma_{22} = 1.$$

If $\gamma_{22} = 0$, the characteristic equation of S_1 has the root α_{22} in the field F, contrary to hypothesis. Hence $\gamma_{22} \neq 0$. Then the transform of S_1 by $\eta_2' = \eta_2 - \lambda\xi_2$, where $\delta_{22} + \lambda\gamma_{22} = 0$, has the form

$$Q = \begin{pmatrix} 0 & \gamma_1 & 0 & 0 \\ -\gamma_1^{-1} & \delta & 1 & 0 \\ 0 & \gamma_1\gamma & \alpha & \gamma \\ 0 & 0 & -\gamma^{-1} & 0 \end{pmatrix}. \tag{56}$$

Its characteristic determinant equals

$$\rho^4 - \rho^3(\alpha + \delta) + \rho^2(2 + \alpha\delta - \gamma_1\gamma) - \rho(\alpha + \delta) + 1. \tag{57}$$

The latter has the factor $\rho^2 - \sigma\rho + 1$ if and only if

$$\sigma^2 - \sigma(\alpha + \delta) + \alpha\delta - \gamma_1\gamma = 0. \tag{58}$$

Assuming here that (58) has a root σ in the field F and that $\rho^2 - \sigma\rho + 1$ is irreducible in F, we determine the conditions under which Q is conjugate with a transformation (55). Let

$$X = a\xi_1 + b\eta_1 + c\xi_2 + d\eta_2, \quad Y = A\xi_1 + B\eta_1 + C\xi_2 + D\eta_2, \tag{33'}$$

where $a, b, \ldots . D$, and τ (below) are elements of F such that

$$\begin{vmatrix} a & b \\ A & B \end{vmatrix} + \begin{vmatrix} c & d \\ C & D \end{vmatrix} = 1. \tag{34'}$$

The conditions that Q shall replace X by $\tau\, Y$, and Y by $-\tau^{-1}X + \sigma Y$, are

$$\tau A = -b\gamma_1^{-1}, \quad \tau B = a\gamma_1 + b\delta + c\gamma_1\gamma, \quad \tau C = b + c\alpha - d\gamma^{-1}, \quad \tau D = c\gamma,$$
$$\sigma A - \tau^{-1}a = -B\gamma_1^{-1}, \quad \sigma B - \tau^{-1}b = A\gamma_1 + B\delta + C\gamma_1\gamma,$$
$$\sigma C - \tau^{-1}c = B + C\alpha - D\gamma^{-1}, \quad \sigma D - \tau^{-1}d = C\gamma.$$

From the first four conditions,

$$b = -\gamma_1\tau A, \quad c = \gamma^{-1}\tau D, \quad a = \gamma_1^{-1}\tau B + \delta\tau A - \tau D, \quad d = \alpha\tau D - \gamma\tau C - \gamma_1\gamma\tau A.$$

In view of these, the last four conditions become

$$\gamma_1\gamma C = (\sigma - \delta) B, \quad D = -(\sigma - \delta) A, \quad B = (\sigma - \alpha) C, \quad \gamma_1\gamma A = -(\sigma - \alpha) D,$$

of which the last two follow from the first two in view of (58), which may be written $(\sigma - \alpha)(\sigma - \delta) = \gamma\gamma_1$. Expressing (34′) in terms of A and B, we get

$$\frac{1}{\tau} = [\gamma_1 + \gamma^{-1}(\sigma - \delta)^2](A^2 + \gamma_1^{-2}B^2) + [2\sigma - \delta + \gamma_1^{-1}\gamma^{-1}\alpha(\sigma - \delta)^2]\, AB$$
$$= \gamma^{-1}(\sigma - \delta)(2\sigma - \alpha - \delta)[A^2 - \sigma(-\gamma_1^{-1}B)\,A + (-\gamma_1^{-1}B)^2].$$

Now $\sigma - \delta \neq 0$. If $2\sigma - \alpha - \delta \neq 0$, the condition merely determines τ whatever values, not both zero, in F be assigned to A and B. Indeed $A^2 - \sigma B_1 A + B_1^2$ is irreducible by the hypothesis. If $2\sigma - \alpha - \delta = 0$, (58) evidently has a double root. *Hence Q is conjugate with* (55) *if its characteristic determinant has in F a factor $\rho^2 - \sigma\rho + 1$, but not its square.*

19. Let next (57) be a perfect square, so that

$$(\alpha - \delta)^2 = -4\gamma\gamma_1. \tag{59}$$

According as F has not or has modulus 2, we have

$$\Delta(\rho) = \{\rho^2 - \tfrac{1}{2}(a + \delta)\rho + 1\}^2, \quad \Delta(\rho) = \{\rho^2 + (\alpha^2 - \gamma\gamma_1)^{\frac{1}{2}} + 1\}^2. \tag{60}$$

Now whatever be the nature of (57), the only function, aside from a constant factor, which Q multiplies by κ, necessarily a root of the characteristic equation, is found to be

$$X_1 \equiv -\gamma\xi_1 + \kappa\gamma\gamma_1\eta_1 + (\kappa^2 - \kappa\delta + 1)\xi_2 + \kappa^{-1}\gamma(\kappa^2 - \kappa\delta + 1)\eta_2. \tag{61}$$

In the present special case, we introduce another function,

$$Y_1 \equiv \gamma\xi_1 + (\kappa^{-2} - 1)\xi_2 + \gamma(\delta - 2\kappa)\eta_2. \tag{62}$$

Denote by X_2, Y_2 the functions derived from X_1 and Y_1 respectively by replacing κ by κ^{-1}. In terms of these variables,* Q becomes, if (59) holds,

$$X_1' = \kappa X_1, \quad Y_1' = \kappa^{-1}Y_1 + \kappa^{-1}X_2, \quad X_2' = \kappa^{-1}X_2, \quad Y_2' = \kappa Y_2 + \kappa X_1. \tag{63}$$

Every transformation in F commutative with Q has the canonical form ($\bar{a}$ being the same function of κ^{-1} that a is of κ);

$$x_1 = aX_1, \quad y_1 = \bar{a}Y_1 + \bar{d}X_2, \quad x_2 = \bar{a}X_2, \quad y_2 = aY_2 + dX_1. \tag{64}$$

Then the transformation from $\xi_1, \eta_1, \xi_2, \eta_2$ to x_1, y_1, x_2, y_2 is the most general transformation T of variables transforming Q into the canonical form (63′), viz., (63) on the variables x_i, y_i. Consider the function

$$\phi_{x,y} \equiv \begin{vmatrix} x_1 & y_1 \\ x_1' & y_1' \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_2' & y_2' \end{vmatrix},$$

---

* The determinant of transformation is found to equal $-\gamma^3\gamma_1(\kappa - \kappa^{-1})^4$. That it is not zero also follows from the Abelian character of T, as shown later.

where x'_i, y'_i are the same linear functions of ξ'_i, η'_i, that x_i, y_i are of ξ_i, η_i. Then, as in §2, $\phi_{x,\,y}$ equals

$$C_{12}\begin{vmatrix} \xi_1 & \eta_1 \\ \xi'_1 & \eta'_1 \end{vmatrix} + C_{13}\begin{vmatrix} \xi_1 & \xi_2 \\ \xi'_1 & \xi'_2 \end{vmatrix} + C_{14}\begin{vmatrix} \xi_1 & \eta_2 \\ \xi'_1 & \eta'_2 \end{vmatrix}$$
$$+ C_{23}\begin{vmatrix} \eta_1 & \xi_2 \\ \eta'_1 & \xi'_2 \end{vmatrix} + C_{24}\begin{vmatrix} \eta_1 & \eta_2 \\ \eta'_1 & \eta'_2 \end{vmatrix} + C_{34}\begin{vmatrix} \xi_2 & \eta_2 \\ \xi'_2 & \eta'_2 \end{vmatrix}.$$

For the transformation T, the value of C_{12} is

$$\begin{vmatrix} -a\gamma & a\varkappa\gamma\gamma_1 \\ \bar{a}\gamma-\bar{d}\gamma & \bar{d}\varkappa^{-1}\gamma\gamma_1 \end{vmatrix} + \begin{vmatrix} -\bar{a}\gamma & \bar{a}\varkappa^{-1}\gamma\gamma_1 \\ a\gamma-d\gamma & d\varkappa\gamma\gamma_1 \end{vmatrix} = -\gamma^2\gamma_1\{A + a\bar{a}\,(\varkappa+\varkappa^{-1})\},$$

where $$A = (\bar{a}d - a\bar{d})(\varkappa-\varkappa^{-1}).$$

Similarly, we find that

$$C_{13} = -\gamma\,\{A\,(\varkappa+\varkappa^{-1}-\delta) + a\bar{a}\,(2\varkappa^2+2\varkappa^{-2}-\delta\,(\varkappa+\varkappa^{-1})\},$$
$$C_{14}=0,\quad C_{23}=0,\quad C_{24}=\gamma\gamma_1\,C_{13},$$
$$C_{34} = -\gamma A\,(\varkappa+\varkappa^{-1}-\delta)^2 + \gamma\,(\varkappa+\varkappa^{-1}-\delta)\,a\bar{a}\,(2+\delta\varkappa+\delta\varkappa^{-1}-3\varkappa^2-3\varkappa^{-2}).$$

The discriminant $\tau \equiv (\varkappa-\varkappa^{-1})^2$ is not zero by hypothesis. Set

$$d/a = \lambda + \varkappa\mu, \text{ whence } \bar{d}/\bar{a} = \lambda + \varkappa^{-1}\mu.$$

Hence

$$A = a\bar{a}\left(\frac{d}{a}-\frac{\bar{d}}{\bar{a}}\right)(\varkappa-\varkappa^{-1}) = a\bar{a}\tau\mu,\quad C_{12} = -\gamma^2\gamma_1\,a\bar{a}\,(\tau\mu+\varkappa+\varkappa^{-1}),$$

$$C_{13} = -\gamma a\bar{a}\,\{\tau\mu\,(\varkappa+\varkappa^{-1}-\delta) + 2\tau + 4 - \delta\,(\varkappa+\varkappa^{-1})\},$$
$$C_{34} = -\gamma\,(\varkappa+\varkappa^{-1}-\delta)\,a\bar{a}\,\{\tau\mu\,(\varkappa+\varkappa^{-1}-\delta) + 3\tau + 4 - \delta\,(\varkappa+\varkappa^{-1})\}.$$

From (57), we may write the characteristic equation

$$\gamma\gamma_1 = (\rho+\rho^{-1}-\alpha)(\rho+\rho^{-1}-\delta). \tag{65}$$

Hence C_{12} may be written

$$C_{12} = -\gamma a\bar{a}\,(\varkappa+\varkappa^{-1}-\alpha)\{\tau\mu\,(\varkappa+\varkappa^{-1}-\delta) + \tau + 4 - \delta\,(\varkappa+\varkappa^{-1})\},$$

By (65), $\varkappa+\varkappa^{-1}-\delta \neq 0$. Hence we can determine μ in the field to make $C_{13}=0$. It then follows immediately that

$$C_{12} = \gamma a\bar{a}\,(\varkappa+\varkappa^{-1}-\alpha)\,\tau,\quad C_{34} = -\gamma a\bar{a}\,(\varkappa+\varkappa^{-1}-\delta)\,\tau.$$

When F has modulus 2, $\alpha=\delta$ and $C_{12}=C_{34}$. If F does not have modulus 2,

$\varkappa + \varkappa^{-1} \equiv \frac{1}{2}(\alpha + \delta)$ by $(60)_1$, whence $C_{13} = C_{34}$. Hence, for any field F, it follows that

$$\phi_{x,y} = C\phi_{\xi,\eta}, \quad C \equiv -\gamma a\bar{a}(\varkappa + \varkappa^{-1} - \delta)\tau. \tag{66}$$

Hence T is a general Abelian transformation (§2) in the enlarged field $F(\varkappa)$. Let Q_1 be a second transformation (56) with the same characteristic determinant (60) as the given Q. It follows from the theory of canonical forms with conjugate variables that Q can be transformed into Q_1 by a general Abelian transformation in F. The latter will be a special Abelian transformation in F if and only if $C_1 = C$. This condition can be satisfied by choice of a and a_1 if the ratio of the two values of $\gamma(\varkappa + \varkappa^{-1} - \delta)$ is of the form $e\bar{e}$, where e belongs to $F(\varkappa)$. If F be the $GF[p^n]$, $e\bar{e} = e^{p^n+1}$ can be made to assume any given value in the $GF[p^n]$. *If* F be the field of all real numbers, $e\bar{e}$ can evidently be made to take any positive value in F, but not a negative value.

THEOREM.—*If two transformations Q have as common characteristic determinant the square of a factor irreducible in F and if F is the $GF[p^n]$, they are conjugate within $SA[4, p^n]$. If F is the field of all real numbers, there are two distinct sets of conjugates.**

Corollary. If F is the $GF[p^n]$, C can be made equal to unity by choice of a, so that Q can be reduced to its canonical form (63) by a special Abelian transformation in the $GF[p^{2n}]$.

20. We now consider Q in the two remaining cases:

(i) $\Delta(\rho)$ irreducible in the field F;

(ii) $\Delta(\rho)$ the product to two irreducible factors $\rho^2 - e\rho + f$, $f \neq 1$.

In either case, the roots of $\Delta(\rho) = 0$ are all distinct and may be designated $\varkappa$, $\varkappa^{-1}$, λ, λ^{-1}. Hence by (57),

$$\varkappa + \varkappa^{-1} + \lambda + \lambda^{-1} = \alpha + \delta. \tag{67}$$

In case (ii), $\varkappa$ and λ are the roots of a quadratic equation irreducible in F. Consider X_1 defined by (61). Denote by Y_1, X_2, Y_2, the functions derived from X_1 upon replacing $\varkappa$ by $\varkappa^{-1}$, λ, λ^{-1}, respectively. In case (i), X_1, Y_1, X_2, Y_2 are conjugate with respect to F. In case (ii) X_1 and X_2 are conjugate with respect to F; likewise, Y_1 and Y_2. In each case the four functions satisfy the requirements as to conjugacy in the general theory of canonical forms,* and Q takes the canonical form

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* For other infinite fields, the question of conjugacy can be determined by introducing the R_{ij} as in the next section.

$$X_1' = \kappa X_1, \quad Y_1' = \kappa^{-1} Y_1, \quad X_2' = \lambda X_2, \quad Y_2' = \lambda^{-1} Y_2. \tag{68}$$

We proceed to prove that the transformation of variables from the ξ_i, η_i to X_i, Y_i satisfies the conditions

$$R_{13} = R_{14} = R_{23} = R_{24} = 0,$$

where R_{ij} is defined in §2. We have

$$R_{13} = \begin{vmatrix} -\gamma & \kappa\gamma\gamma_1 \\ -\gamma & \lambda\gamma\gamma_1 \end{vmatrix} + \begin{vmatrix} \kappa^2 - \kappa\delta + 1 & \kappa^{-1}\gamma(\kappa^2 - \kappa\delta + 1) \\ \lambda^2 - \lambda\delta + 1 & \lambda^{-1}\gamma(\lambda^2 - \lambda\delta + 1) \end{vmatrix}$$
$$= \gamma(\kappa - \lambda)\{\gamma\gamma_1 + (\kappa + \kappa^{-1} - \delta)(\lambda + \lambda^{-1} - \delta)\}.$$

Eliminating $\gamma\gamma_1$ by means of (65) for $\rho = \kappa$, we get

$$R_{13} = \gamma(\kappa - \lambda)(\kappa + \kappa^{-1} - \delta)(\kappa + \kappa^{-1} + \lambda + \lambda^{-1} - \alpha - \delta).$$

Hence $R_{13} = 0$ by (67). Replacing λ by λ^{-1}, we get $R_{14} = 0$. Replacing κ by κ^{-1}, we get $R_{23} = 0$. Making both replacements, we get $R_{24} = 0$. Replacing λ by κ^{-1}, we get

$$R_{12} = \gamma(\kappa - \kappa^{-1})\, C, \quad C \equiv (\kappa + \kappa^{-1} - \delta)(2\kappa + 2\kappa^{-1} - \alpha - \delta). \tag{69}$$

Replacing κ by λ in the preceding, we get

$$R_{34} = \gamma(\lambda - \lambda^{-1})\, C_1, \quad C_1 \equiv (\lambda + \lambda^{-1} - \delta)(2\lambda + 2\lambda^{-1} - \alpha - \delta). \tag{70}$$

In either case (i) or case (ii), we have $C \neq 0$, $C_1 \neq 0$.

For case (ii), we introduce in place of X_1, X_2 the variables

$$x_1 = X_1 \div \gamma(\kappa - \kappa^{-1})\, C, \quad x_2 = X_2 \div \gamma(\lambda - \lambda^{-1})\, C_1.$$

Then x_1 and x_2 are conjugate with respect to F, and (68) becomes

$$x_1' = \kappa x_1, \quad Y_1' = \kappa^{-1} Y_1, \quad x_2' = \lambda x_2, \quad Y_2' = \lambda^{-1} Y_2. \tag{68'}$$

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* American Journal, Vol. XXIV, pp. 101–108.

For the transformation of variables from ξ_i, η_i to x_i, Y_i, we have

$$R'_{13} = R'_{14} = R'_{23} = R'_{24} = 0, \quad R'_{12} = 1, \quad R'_{34} = 1,$$

so that it is a special Abelian transformation in the extended field $F(\kappa, \lambda)$.

THEOREM.—*Within the group $SA(4, F)$ two transformations are conjugate if they have the same characteristic determinant which is the product of two irreducible factors $\rho^2 - a\rho + b$, $b \neq 1$.*

For case (i), we introduce in place of X_1, Y_1, X_2, Y_2 the variables*

$$x_1 = X_1/f(\kappa), \quad y_1 = Y_1/f(\kappa^{-1}), \quad x_2 = X_2/f(\lambda), \quad y_2 = Y_2/(\lambda^{-1}),$$

where $f(\rho) = a\rho^3 + b\rho^2 + c\rho + d$, a, b, c, d being undetermined elements of F not all zero. Then x_1, y_1, x_2, y_2 are conjugate with respect to F and (68) becomes

$$x'_1 = \kappa x_1, \quad y'_1 = \kappa^{-1} y_1, \quad x'_2 = \lambda x_2, \quad y'_2 = \lambda^{-1} y_2. \tag{68''}$$

Let T denote the transformation of variables from ξ_i, η_i to x_i, y_i. For it,

$$R''_{13} = R''_{14} = R''_{23} = R''_{24} = 0, \quad R''_{12} = \frac{\gamma(\kappa - \kappa^{-1})C}{f(\kappa)f(\kappa^{-1})}, \quad R''_{34} = \frac{\gamma(\lambda - \lambda^{-1})C_1}{f(\lambda)f(\lambda^{-1})}.$$

For brevity, set $R''_{12} = \phi(\kappa)$. Then $R''_{24} = \phi(\lambda)$. By the Corollary to §3,

$$\begin{vmatrix} \xi_1 & \eta_1 \\ \xi_1^* & \eta_1^* \end{vmatrix} + \begin{vmatrix} \xi_2 & \eta_2 \\ \xi_2^* & \eta_2^* \end{vmatrix} = \frac{1}{\phi(\kappa)} \begin{vmatrix} x_1 & y_1 \\ x_1^* & y_1^* \end{vmatrix} + \frac{1}{\phi(\lambda)} \begin{vmatrix} x_2 & y_2 \\ x_2^* & y_2^* \end{vmatrix},$$

where x_i^*, y_i^* are the same functions of ξ_i^*, η_i^* that x_i, y_i are of ξ_i, η_i, so that T affects the two sets of variables cogrediently. It follows that two transformations Q, with the same irreducible characteristic determinant $\Delta(\rho)$, are conjugate within $SA(4, F)$, if (and, at least for normal fields,* only if) it be possible to choose $f(\rho)$ so that $\phi(\kappa)$ shall depend only upon the invariant $\Delta(\rho)$ and not upon the individual coefficients of Q. Now

$$\phi(\kappa) = (\kappa - \kappa^{-1})(2\kappa + 2\kappa^{-1} - \alpha - \delta)D, \quad D \equiv \gamma(\kappa + \kappa^{-1} - \delta)/f(\kappa)f(\kappa^{-1}).$$

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* The field $F(\kappa, \lambda)$ may be a normal field, i. e., $\lambda =$ rat. func. (κ), with coefficients in F. This is the case when F is the $GF[p^n]$. It would be interesting to determine, for $F(\kappa, \lambda)$ not normal, whether or not it is possible to retain the advantage of conjugate variables and yet employ $f(\kappa, \lambda)$ in place of $f(\kappa)$. By (67), $\lambda^2 - \lambda(\alpha + \delta - \kappa - \kappa^{-1}) + 1 = 0$, so that $f(\kappa, \lambda)$ may be written $f_1(\kappa) + \lambda f_2(\kappa)$.

When F is the $GF[p^n]$, $\varkappa^{-1}$ must equal $\varkappa^{p^n}$, $\varkappa^{p^{2n}}$ or $\varkappa^{p^{3n}}$, since these three together with $\varkappa$ furnish the distinct roots of $\Delta(\rho)=0$. But $\varkappa^{-1}=\varkappa^{p^n}$ would require that $\varkappa^{p^{2n}}=\varkappa^{-p^n}=\varkappa$. Again $\varkappa^{-1}=\varkappa^{p^{3n}}$ would require that $\varkappa^{-p^n}=\varkappa^{p^{4n}}=\varkappa$. Hence must $\varkappa^{-1}=\varkappa^{p^{2n}}$, so that

$$f(\varkappa^{-1})=[f(\varkappa)]^{p^{2n}}, \quad K\equiv\varkappa+\varkappa^{-1}=\text{element of } GF[p^{2n}].$$

To make $D=1$, it suffices to take as $f(\varkappa)$ a root of

$$f^{p^{2n}+1}=\gamma(K-\delta),$$

all of whose roots f belong to the $GF[p^{4n}]$. Indeed, the power $p^{2n}-1$ of the second member is unity, not being zero in view of the irreducibility of $\Delta(\rho)$. *Hence two transformations Q having the same irreducible $\Delta(\rho)$ are conjugate within $SA(4, p^n)$.*

For a general field F, we have, by (65),

$$K^2=\varepsilon K+\tau, \quad \varepsilon\equiv\alpha+\delta, \quad \tau=\gamma\gamma_1-\alpha\delta\neq 0, \quad K=\varkappa+\varkappa^{-1}.$$

Then $f(\varkappa)f(\varkappa^{-1})$ takes the form $rK+s$, where

$$\begin{cases} r=ab+bc+cd+(ac+bd)\varepsilon+ad(\varepsilon^2+\tau-3), \\ s=a^2+b^2+c^2+d^2+(ac+bd)(\tau-2)+ad\varepsilon\tau. \end{cases} \tag{71}$$

Hence

$$\frac{1}{D}=\frac{rK+s}{\gamma(K-\delta)}=\frac{(rK+s)(K-\alpha)}{\gamma^2\gamma_1}=\frac{K(s+r\delta)+r\tau-s\alpha}{\gamma^2\gamma_1}.$$

Then $D=1$ if and only if $r=\gamma$, $s=-\delta\gamma$. Hence any two Q with the same $\Delta(\rho)$ are conjugate if equations (71) are solvable in F for a, b, c, d, whatever values r and s have in F. They are equivalent to

$$P\{ac+bd+(\varepsilon+2)ad\}+(a+b+c+d)^2=s+2r,$$
$$N\{ac+bd+(\varepsilon-2)ad\}+(a-b+c-d)^2=s-2r,$$

where

$$P=\tau-4+2\varepsilon=\gamma\gamma_1-(2-\alpha)(2-\delta),$$
$$N=\tau-4-2\varepsilon=\gamma\gamma_1-(-2-\alpha)(-2-\delta)$$

are not zero in view of the irreducibility of $\gamma\gamma_1-(K-\alpha)(K-\delta)$.

We may attack the problem otherwise. If Q and Q' have the same $\Delta(\rho)$ they are conjugate if $D = D'$, viz., if

$$s' + r'\delta' = \mu(s + r\delta), \quad r'\tau - s'\alpha' = \mu(r\tau - s\alpha), \quad \mu = \gamma'^2\gamma_1'/\gamma^2\gamma_1,$$

for suitable values of $a, b, c, d, a', b', c', d'$ in F. We thus have two quadratic conditions on 8 unknown quantities.

21. We next consider transformation S_1 of §17 for $m = 3$. Applying certain Abelian conditions, we have

$$S_1 = \begin{pmatrix} 0 & \gamma & 0 & 0 & 0 & 0 \\ -\gamma^{-1} & \delta & 1 & 0 & 0 & 0 \\ 0 & \gamma\gamma_{22} & \alpha_{22} & \gamma_{22} & \alpha_{23} & \gamma_{23} \\ 0 & \gamma\delta_{22} & \beta_{22} & \delta_{22} & \beta_{23} & \delta_{23} \\ 0 & \gamma\gamma_{32} & \alpha_{32} & \gamma_{32} & \alpha_{33} & \gamma_{33} \\ 0 & \gamma\delta_{32} & \beta_{32} & \delta_{32} & \beta_{33} & \delta_{33} \end{pmatrix}. \tag{72}$$

If $\gamma_{32} = \delta_{32} = 0$, we may transform S_1 by a transformation on ξ_3 and η_3 and obtain a like transformation with also $\gamma_{23} = 0$. If γ_{32} and δ_{32} are not both zero, we may transform S_1 by a transformation on ξ_3 and η_3 and make $\gamma_{32} = 0$, $\delta_{32} \neq 0$. We therefore treat S_1 under two cases:

$$\text{(I).} \quad \gamma_{32} = \delta_{32} = \gamma_{23} = 0; \qquad \text{(II).} \quad \gamma_{32} = 0, \quad \delta_{32} \neq 0.$$

Case I. If also $\gamma_{22} = 0$, the Abelian conditions C_{34} and C_{45} give $\alpha_{22}\delta_{22} = 1$, $\delta_{22}\alpha_{23} = 0$, whence $\xi_2' = \alpha_{22}\xi_2$, so that $\Delta(\rho) = 0$ has a root α_{22} in F. Hence $\gamma_{22} \neq 0$. Then Abelian condition C_{46} gives $\delta_{23} = 0$. Transforming S_1 by $\eta_2' = \eta_2 + \lambda\xi_2$, $\delta_{22} + \lambda\gamma_{22} = 0$, we obtain a transformation S_1' of the form S_1 with $\gamma_{32} = \delta_{32} = \gamma_{23} = 0$, $\delta_{23} = 0$, $\gamma_{22} \neq 0$, and also $\delta_{22} = 0$. Then Abelian conditions C_{45} and R_{35} give $\beta_{23} = 0$, $\alpha_{23}\gamma_{33} - \gamma_{22}\alpha_{32} = 0$. If $\gamma_{33} = 0$, then $\alpha_{32} = 0$, $\xi_3' = \alpha_{33}\xi_3$, and $\Delta(\rho) = 0$ has a root in F. Hence $\gamma_{33} \neq 0$. Transforming S_1' by $\eta_3' = \eta_3 + \lambda\xi_3$, $\delta_{33} + \lambda\gamma_{33} = 0$, we obtain a like transformation with also $\delta_{33} = 0$. Then $\beta_{32} = 0$ by Abelian condition R_{36}. The resulting Abelian transformation is

$$\begin{pmatrix} 0 & \gamma & 0 & 0 & 0 & 0 \\ -\gamma^{-1} & \delta & 1 & 0 & 0 & 0 \\ 0 & \gamma\gamma_{22} & \alpha_{22} & \gamma_{22} & \alpha_{23} & 0 \\ 0 & 0 & -\gamma_{22}^{-1} & 0 & 0 & 0 \\ 0 & 0 & \gamma_{22}^{-1}\alpha_{23}\gamma_{33} & 0 & \alpha_{33} & \gamma_{33} \\ 0 & 0 & 0 & 0 & -\gamma_{33}^{-1} & 0 \end{pmatrix}.$$

If $\alpha_{23}=0$, its transform by M_3P_{13} is of the form (55). If $\alpha_{23}\neq 0$, its transform by $T_{3,\alpha_{23}}$ is of like form with $\alpha_{23}=1$, viz., (92) below.

Case II. If $\gamma_{22}\neq 0$, the transform of S_1 by $R_{2,3,\lambda}$, $\delta_{32}-\lambda\gamma_{22}=0$, is of the form S_1 with $\gamma_{32}=\delta_{32}=0$, and hence falls under case I. Let next $\gamma_{22}=0$. Transforming by $T_{3,\delta_{23}}$ we obtain an S_1 with $\gamma_{32}=\gamma_{22}=0$, $\delta_{32}=1$. Transforming the latter by $Q_{3,2,\delta_{22}}$, we obtain a like transformation S' with also $\delta_{22}=0$. Then Abelian conditions C_{34}, C_{45}, C_{46} give $\alpha_{32}=1$, $\alpha_{33}=0$, $\gamma_{33}=0$.

If S' has $\gamma_{23}\neq 0$, its transform by

$$\eta_2'=\eta_2+\lambda\xi_2,\quad \eta_3'=\eta_3+\mu\xi_3,\quad \alpha_{23}-\mu\gamma_{23}=0,\quad \delta_{23}+\lambda\gamma_{23}=0,$$

is of the form S' with $\alpha_{23}=\delta_{23}=0$. Then $R_{2,3,\beta_{22}}$, transforms the latter into

$$\begin{pmatrix} 0 & \gamma & 0 & 0 & 0 & 0 \\ -\gamma^{-1} & \delta & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & g \\ 0 & 0 & 0 & 0 & -g^{-1} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \gamma & \beta & 1 & g^{-1}\alpha & 0 \end{pmatrix} \tag{73}$$

which is studied in §§24–26. Its characteristic determinant is

$$\rho^6-(\rho^5+\rho)(\delta+\alpha)+(\rho^4+\rho^2)(1+\delta\alpha-\beta g)-\rho^3(2\alpha+\gamma g-\delta\beta g)+1. \tag{74}$$

Let next $\gamma_{23}=0$ in S'. Then $\alpha_{23}\delta_{23}=1$, $\alpha_{22}+\alpha_{23}\delta_{33}=0$ by Abelian conditions. If $\alpha_{22}\neq 0$, we transform by $\eta_2'=\eta_2+\lambda\xi_2$, $\eta_3'=\eta_3+\mu\xi_3$, where $\beta_{22}+\lambda\alpha_{22}=0$, $\beta_{23}+\lambda\alpha_{23}-\mu\delta_{23}=0$, and obtain a transformation which in view of the Abelian conditions may be written

$$\begin{pmatrix} 0 & \gamma & 0 & 0 & 0 & 0 \\ -\gamma^{-1} & \delta & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & g^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & g \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \gamma & \beta & 1 & 0 & -g\alpha \end{pmatrix}. \tag{75}$$

It is studied in §23. Its characteristic determinant is

$$\rho^6-(\rho^5+\rho)(\delta+\alpha-g\alpha)+(\rho^4+\rho^2)(1+\delta\alpha-g\delta\alpha-g-g^{-1}-g\alpha^2)$$
$$-\rho^3(2\alpha-2g\alpha-\delta g-\delta g^{-1}-g\delta\alpha^2)+1. \tag{76}$$

If $\alpha_{22} = 0$, then $\delta_{33} = 0$, $\beta_{22} = \delta_{23}\beta_{33}$. Transforming by $\eta_3' = \eta_3 - \beta_{32}\xi_3$, we obtain

$$\begin{pmatrix} 0 & \gamma & 0 & 0 & 0 & 0 \\ -\gamma^{-1} & \delta & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & \beta & 0 & \beta' & \alpha^{-1} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 1 & \alpha\beta & 0 \end{pmatrix}. \tag{77}$$

Its characteristic determinant is seen to equal

$$(\rho^2 - \rho\delta + 1)(\rho^2 - \alpha)(\rho^2 - \alpha^{-1}). \tag{78}$$

22. The last result suggests that we determine the conditions under which (77) is conjugate within G with a transformation (55) which replaces ξ_1 by $\tau\eta_1$ and η_1 by $-\tau^{-1}\xi_1 + \delta\eta_1$. The conditions that (77) shall replace X and Y, defined by (33), by respectively τY and $-\tau^{-1}X + \delta Y$, are seen to be

$$\tau A = -b\gamma^{-1}, \quad \tau B = a\gamma + b\delta + j\gamma, \quad \tau C = b + e + d\beta,$$
$$\tau D = j, \quad \tau E = c\alpha + d\beta' + j\alpha\beta, \quad \tau J = d\alpha^{-1},$$
$$\delta A - \tau^{-1}a = -B\gamma^{-1}, \quad \delta B - \tau^{-1}b = A\gamma + B\delta + J\gamma,$$
$$\delta C - \tau^{-1}c = B + E + D\beta,$$
$$\delta D - \tau^{-1}d = J, \quad \delta E - \tau^{-1}e = C\alpha + D\beta' + J\alpha\beta \quad \delta J - \tau^{-1}j = D\alpha^{-1}.$$

From the first set of six conditions, we get

$$b = -\gamma\tau A, \quad j = \tau D, \quad d = \alpha\tau J, \quad e = \tau C + \gamma\tau A - \alpha\beta\tau J,$$
$$a = \gamma^{-1}\tau B + \delta\tau A - \tau D, \quad c = \alpha^{-1}\tau E - \beta'\tau J - \beta\tau D.$$

Substituting these values in the second set of six, we get

$$D = 0, \quad J = 0, \quad B = \delta C - E - \alpha^{-1}E, \quad \gamma A = \delta E - C - \alpha C.$$

Hence $d = j = 0$. Abelian condition (34) thus becomes

$$aB - bA = 1, \quad \text{or} \quad \gamma^{-1}\tau(B^2 + \gamma\delta AB + \gamma^2A^2) = 1. \tag{79}$$

Now C and E can be chosen in the field to make $B \neq 0$, unless $\delta = 0$, $\alpha = -1$ (when also $A \equiv 0$). For $B \neq 0$, the condition may be written

$$\gamma^{-1}\tau B^2[1 - \delta(-\gamma AB^{-1}) + (-\gamma AB^{-1})^2] = 1. \tag{79'}$$

This merely serves to determine τ in F for any given values of A and B ($B \neq 0$)

in F, since $1-\delta\rho+\rho^2$ is irreducible in F by hypothesis. *Hence* (77) *is conjugate within* G *with a transformation* (55) *if, and only if,* δ *and* $\alpha+1$ *are not both zero.*

It remains to consider (77) in the case $\delta=0$, $\alpha=-1$, whence (78) becomes $(\rho^2+1)^3$. Since ρ^2+1 shall be irreducible, F cannot have modulus 2. Then $R_{2,3,-\beta/2}$ transforms (77) into a similar transformation with $\beta=\delta=0$, $\alpha=-1$. Now $L'_{3,-\beta'}$ transforms the latter into (75) for $\delta=\alpha=0$, $g=-1$, $\beta=-\beta'$.

23. For type (75), the characteristic determinant (76) factors into

$$(\rho^2-\rho\delta+1)(\rho^2-\alpha\rho-g^{-1})(\rho^2+g\alpha\rho-g). \tag{76'}$$

The conditions that (75) shall replace X and Y, defined by (33), by respectively τY and $-\tau^{-1}X+\delta Y$ are seen (as in §22) to reduce to

$$\begin{aligned}
&b=-\gamma\tau A,\quad j=\tau D,\quad c=g\tau E,\quad a=\gamma^{-1}\tau B+\delta\tau A-\tau D,\\
&d=g^{-1}\tau J+\alpha\tau D,\quad e=\tau C+\gamma\tau A-g\alpha\tau E-\beta\tau D;\\
&D=0,\quad J=0,\quad B=C(\delta-\alpha)-E(g+1),\quad \gamma A=-C(g^{-1}+1)+E(\delta+g\alpha).
\end{aligned}$$

The Abelian condition (34) reduces to (79). Now C and E can be chosen in F to make $B\neq 0$ unless $\delta=\alpha$, $g=-1$ (when also $A\equiv 0$). *Hence* (75) *is conjugate within* G *with a transformation* (55) *if, and only if,* $\delta-\alpha$ *and* $g+1$ *are not both zero.*

24. Now $\rho^2-\sigma\rho+1$ is a factor of (74) if, and only if,

$$\chi(\sigma)\equiv\sigma^3-\sigma^2(\alpha+\delta)+\sigma(\alpha\delta-\beta g-2)+2\delta-\gamma g+\beta\delta g=0. \tag{80}$$

Supposing this to have a root σ in the field F, and that $\rho^2-\sigma\rho+1$ is irreducible, we determine the conditions under which (73) is conjugate with a transformation (55). Let X and Y be defined by (33). The conditions that (73) shall replace X by τY and Y by $-\tau^{-1}X+\sigma Y$ are

$$\begin{aligned}
&\tau A=-b\gamma^{-1},\quad \tau B=a\gamma+b\delta+j\gamma,\quad \tau C=b+c\alpha+e+j\beta,\quad \tau D=j,\\
&\tau E=-g^{-1}d+g^{-1}\alpha j,\quad \tau J=cg,\quad \sigma A-\sigma^{-1}a=-B\gamma^{-1},\\
&\sigma B-\tau^{-1}b=A\gamma+B\delta+J\gamma,\quad \sigma C-\tau^{-1}c=B+C\alpha+E+J\beta,\\
&\sigma D-\tau^{-1}d=J,\quad \sigma E-\tau^{-1}e=-g^{-1}D+g^{-1}\alpha J,\quad \sigma J-\tau^{-1}j=Cg.
\end{aligned}$$

From the first six conditions,

$$\left.\begin{aligned} &b=-\gamma\tau A,\quad j=\tau D,\quad c=g^{-1}\tau J,\quad a=\gamma^{-1}\tau B+\delta\tau A-\tau D,\\ &d=-g\tau E+\alpha\tau D,\quad e=\tau C+\gamma\tau A-g^{-1}\alpha\tau J-\beta\tau D.\end{aligned}\right\}\qquad(81)$$

Substituting these values in the final six conditions, we get

$$\left.\begin{aligned} D=-(\sigma-\delta)A,\quad J=\gamma^{-1}(\sigma-\delta)B,\quad C&=g^{-1}(\sigma-\delta)(\gamma^{-1}\sigma B+A),\\ E&=g^{-1}(\sigma-\delta)(\gamma^{-1}B+\sigma A-\alpha A),\\ E=g^{-1}(\sigma-\alpha)(\sigma-\delta)(\gamma^{-1}\sigma B+A)&-(g^{-1}+\beta)(\sigma-\delta)\gamma^{-1}B-B,\\ \sigma E=g^{-1}(\sigma-\delta)(\gamma^{-1}\sigma B+A)&+(g^{-1}+\beta)(\sigma-\delta)A+\gamma A.\end{aligned}\right\}\qquad(82)$$

The difference of the first two expressions for E vanishes in view of (80). Multiplying the second expression for E by σ and subtracting the given expression for σE, we obtain $\gamma^{-1}\sigma B+A$ times (80). Hence the three values for E are consistent. In view of (82), (81) become

$$\left.\begin{aligned} &b=-\gamma\tau A,\quad c=g^{-1}\gamma^{-1}(\sigma-\delta)\tau B,\quad j=-(\sigma-\delta)\tau A,\\ &a=\gamma^{-1}\tau B+\sigma\tau A,\quad d=-\gamma^{-1}(\sigma-\delta)\tau B-\sigma(\sigma-\delta)\tau A,\\ &e=[\gamma+(\beta+g^{-1})(\sigma-\delta)]\tau A+g^{-1}\gamma^{-1}(\sigma-\alpha)(\sigma-\delta)\tau B.\end{aligned}\right\}\qquad(83)$$

Then (34) becomes,* on substitution of the values from (82) and (83),

$$\frac{1}{\tau}=M(A^2+\sigma AB\gamma^{-1}+B^2\gamma^{-2}),\quad M\equiv\gamma+(2g^{-1}\sigma-g^{-1}\alpha)(\sigma-\delta)^2.$$

If $M\neq 0$, this equation determines τ, whatever be the values of A and B, not both zero (see §22). Eliminating γ by means of (80),

$$M=g^{-1}(\sigma-\delta)(3\sigma^2-2\sigma\alpha-2\sigma\delta+\alpha\delta-2-\beta g).$$

The final factor is the derivative of the left member of (80). Also, $\sigma\neq\delta$ in view of (80). Hence $M=0$ if, and only if, σ is a double root of (80). In this case, the third root of (80) may be determined by rational process, and hence belongs to F. If it differs from σ, it is a simple root and hence leads to an $M\neq 0$.

THEOREM.—*If* (74) *has in the field an irreducible factor* $\rho^2-\sigma\rho+1$ *and differs from* $(\rho^2-\sigma\rho+1)^3$, *then* (73) *is conjugate within* G *with a transformation* (55).

---

* The coefficient of $\gamma^{-1}AB$ is initially $\gamma(2\sigma-\delta)+(\beta+2g^{-1}+g^{-1}\sigma^2)(\sigma-\delta)^2$. Replacing $\beta g(\sigma-\delta)$ by its value $\sigma^3-\sigma^2(\alpha+\delta)+\sigma(\alpha\delta-2)+2\delta-\gamma g$ from (80), we get σM.

25. Suppose that the characteristic determinant of a special Abelian transformation has in the field F neither a linear factor nor a quadratic factor of the form $\rho^2-\sigma\rho+1$. It therefore has no quadratic factor whatever. For if $\rho^2-e\rho+f$, $f\neq 1$, be a factor then, by §5, $\rho^2-\frac{e}{f}\rho+\frac{1}{f}$ would be a factor, so that there would be a third factor of the form $\rho^2-\sigma\rho+1$, with σ in F. If an irreducible cubic factor vanishes for $\varkappa$, λ, μ, there occurs a second irreducible cubic factor vanishing for $\varkappa^{-1}$, λ^{-1}, μ^{-1} and the six quantities

$$\varkappa,\ \lambda,\ \mu,\quad \varkappa^{-1},\ \lambda^{-1},\ \mu^{-1} \tag{84}$$

are all distinct. Thus, if $\varkappa^{-1}=\lambda$, then μ would belong to F. Finally, if $\Delta(\rho)$ be irreducible, it vanishes for 6 distinct values (84).

26. In view of §§24 and 25, it remains to discuss (73) when (74) vanishes for 6 distinct values (84). Now (73) replaces

$$X_1\equiv a\xi_1+b\eta_1+c\xi_2+d\eta_2+e\xi_3+j\eta_3$$

by $\varkappa X_1$ if, and only if,

$$\varkappa a=-b\gamma^{-1},\quad \varkappa b=a\gamma+b\delta+j\gamma,\quad \varkappa c=b+c\alpha+e+\beta j,$$
$$\varkappa d=j,\quad \varkappa e=-g^{-1}d+g^{-1}\alpha j,\quad \varkappa j=cg.$$

Now $a\neq 0$, since $a=0$ requires that $X_1\equiv 0$. For $a=1$, we have

$$X_1=\xi_1-\gamma\varkappa\eta_1+g^{-1}\varkappa D\xi_2+\varkappa^{-1}D\eta_2+g^{-1}\varkappa^{-1}(\alpha-\varkappa^{-1})D\xi_3+D\eta_3, \tag{85}$$

where $D\equiv -\varkappa^2+\varkappa\delta-1$, and $\varkappa$ is a root of (74). Let Y_1, X_2, Y_2, X_3, Y_3 be the functions derived from (85) upon replacing $\varkappa$ by $\varkappa^{-1}$, λ, λ^{-1}, μ, μ^{-1}, respectively. Let $D'\equiv -\lambda^2+\lambda\delta-1$. Then

$$\begin{aligned}
R_{13}&=\begin{vmatrix}1 & -\gamma\varkappa\\ 1 & -\gamma\lambda\end{vmatrix}+\begin{vmatrix}g^{-1}\varkappa D & \varkappa^{-1}D\\ g^{-1}\lambda D' & \lambda^{-1}D'\end{vmatrix}+\begin{vmatrix}g^{-1}\varkappa^{-1}(\alpha-\varkappa^{-1})D & D\\ g^{-1}\lambda^{-1}(\alpha-\lambda^{-1})D' & D'\end{vmatrix}\\
&=(\varkappa-\lambda)\{\gamma+g^{-1}(\delta-\varkappa-\varkappa^{-1})(\delta-\lambda-\lambda^{-1})(\varkappa+\lambda)\\
&\qquad\qquad +g^{-1}(\delta-\varkappa-\varkappa^{-1})(\delta-\lambda-\lambda^{-1})(\varkappa^{-1}+\lambda^{-1}-\alpha)\},\\
R_{13}&=g^{-1}(\varkappa-\lambda)\{g\gamma\\
&\qquad\qquad +(\delta-\varkappa-\varkappa^{-1})(\delta-\lambda-\lambda^{-1})(\varkappa+\varkappa^{-1}+\lambda+\lambda^{-1}-\alpha)\}.
\end{aligned} \tag{86}$$

In view of the notation (84) for the roots of (74), we have

$$\varkappa+\varkappa^{-1}+\lambda+\lambda^{-1}+\mu+\mu^{-1}=\alpha+\delta. \tag{87}$$

Hence

$$R_{13} = g^{-1}(\kappa - \lambda)\{g\gamma + (\delta - \kappa - \kappa^{-1})(\delta - \lambda - \lambda^{-1})(\delta - \mu - \mu^{-1})\}.$$

Now $\kappa + \kappa^{-1}$, $\lambda + \lambda^{-1}$, $\mu + \mu^{-1}$are the roots of (80). Hence

$$(\delta - \kappa - \kappa^{-1})(\delta - \lambda - \lambda^{-1})(\delta - \mu - \mu^{-1}) = \chi(\delta), \quad \gamma g + \chi(\delta) \equiv 0. \quad R_{13} = 0.$$

Replacing κ, λ by any two values (84), not inverse to each other,

$$R_{ij} = 0, \tag{88}$$

$$[i, j = 1, \ldots, 6, \text{ except for } (i, j) = (1, 2), (3, 4), (5, 6)].$$

Replacing λ by κ^{-1} in (86), we get*

$$R_{12} = g^{-1}(\kappa - \kappa^{-1})\{g\gamma + (\delta - \kappa - \kappa^{-1})^2(2\kappa + 2\kappa^{-1} - \alpha)\}. \tag{89}$$

Let $R_{12} = r(\kappa)$. Then $R_{34} = r(\lambda)$, $R_{56} = r(\mu)$.

Let first $\Delta(\rho)$ be the product of two irreducible cubic factors in F, the first with the roots κ, λ, μ, the second with the roots κ^{-1}, λ^{-1}, μ^{-1} (§25). Let

$$x_1 = X_1/r(\kappa), \quad x_2 = X_2/r(\lambda), \quad x_3 = X_3/r(\mu),$$

so that x_1, x_2, x_3 are conjugate with respect to F; likewise Y_1, Y_2 Y_3. Then (73) takes the canonical form

$$x_1' = \kappa x_1, \quad Y_1' = \kappa^{-1}Y_1, \quad x_2 = \lambda x_2, \quad Y_2' = \lambda^{-1}Y_2, \quad x_3' = \mu x_3, \quad Y_3 = \mu^{-1}Y_3. \tag{90}$$

For the transformation of variables from ξ_i, η_i to x_i, Y_i, we have

$$R_{12}' = R_{34}' = R_{56}' = 1, \quad R_{ij}' = 0 \text{ [except for } (i, j) = (1, 2), (3, 4), (5, 6)],$$

so that it is a special Abelian transformation in the field $F(\kappa, \lambda, \mu)$.

THEOREM.—*Within the group $SA(6, F)$, two transformations (73) are conjugate if they have as common characteristic determinant the product of two irreducible cubic factors.*

Let next $\Delta(\rho)$ be irreducible in F. Let

$$x_1 = X_1/f(\kappa), \quad y_1 = Y_1/f(\kappa^{-1}), \ldots, x_3 = X_3/f(\mu), \quad y_3 = Y_3/f(\mu^{-1}),$$

where $f(\rho)$ is a polynomial in ρ with coefficients in F. Then

$$x_1' = \kappa x_1, \quad y_1' = \kappa^{-1}y_1, \quad x_2' = \lambda x_2, \quad y_2' = \lambda^{-1}y_2, \quad x_3' = \mu x_3, \quad y_3' = \mu^{-1}y_3. \tag{91}$$

---

* To show that $R_{12} \mp 0$, we note that (74) gives, for $\rho = \kappa$,

$$g\gamma = (\kappa + \kappa^{-1} - \delta)[\kappa^2 + \kappa^{-2} - a(\kappa + \kappa^{-1}) - g\beta],$$

$$\therefore R_{12} = g^{-1}(\kappa - \kappa^{-1})(\delta - \kappa - \kappa^{-1})\{-3\kappa^2 - 3\kappa^{-2} + 2(a + \delta)(\kappa + \kappa^{-1}) + g\beta - a\delta - 4\}.$$

But $\Delta(\rho)$ has no quadratic or quartic factor in F.

Let $\phi(x) = R_{12}/f(x)f(x^{-1})$. Then, as in §20,

$$\sum_{i=1}^{3} \begin{vmatrix} \xi_i & \eta_i \\ \xi_i^* & \eta_i^* \end{vmatrix} = \frac{1}{\phi(x)} \begin{vmatrix} x_1 & y_1 \\ x_1^* & y_1^* \end{vmatrix} + \frac{1}{\phi(\lambda)} \begin{vmatrix} x_2 & y_2 \\ x_2^* & y_2^* \end{vmatrix} + \frac{1}{\phi(\mu)} \begin{vmatrix} x_3 & y_3 \\ x_3^* & y_3^* \end{vmatrix}.$$

Let $x + x^{-1} = K$. Then $\phi(x) = (x - x^{-1})\,\psi(x)$, where

$$\psi(x) = g^{-1}\{g\gamma + (\delta - K)^2(2K - \alpha)\}/f(x)f(x^{-1}).$$

Further discussion is limited to the case when F is the $GF[p^n]$. Then $\Delta(\rho) = 0$ has the roots

$$x,\ x^{p^n},\ x^{p^{2n}},\ x^{p^{3n}},\ x^{p^{4n}},\ x^{p^{5n}} \qquad (x^{p^{6n}} = x).$$

Among these consequently occurs x^{-1}. Now $x^{-1} = x^{p^n}$ would give

$$x^{p^{2n}} = x^{-p^n} = x;$$

$x^{-1} = x^{p^{2n}}$ would give $x^{p^{4n}} = x$; $x^{-1} = x^{p^{4n}}$ would give $x = x^{p^{6n}} = x^{-p^{2n}}$, whence $x^{p^{4n}} = x$. Also $x^{-1} \neq x$, $x^{-1} \neq x^{p^{5n}}$. Hence $x^{-1} = x^{p^{3n}}$. Hence

$$f(x^{-1}) = [f(x)]^{p^{3n}}, \quad K = x + x^{-1} = \text{element of } GF[p^{3n}].$$

To make $\psi(x) = 1$, it suffices to take as $f(x)$ a root of

$$f^{p^{3n}+1} = \gamma + g^{-1}(\delta - K)^2(2K - \alpha),$$

all of whose roots f belong to the $GF[p^{6n}]$. Indeed, the second member is a mark $\neq 0$ of the $GF[p^{3n}]$, so that the power $p^{3n} - 1$ of it equals unity.

THEOREM.—*Within $SA(6, p^n)$, two transformations (73) having the same irreducible characteristic determinant are conjugate.*

27. Consider finally the type, obtained at the end of Case I, §21,

$$\begin{pmatrix} 0 & \gamma & 0 & 0 & 0 & 0 \\ -\gamma^{-1} & \delta & 1 & 0 & 0 & 0 \\ 0 & \gamma g & \alpha & g & 1 & 0 \\ 0 & 0 & -g^{-1} & 0 & 0 & 0 \\ 0 & 0 & g^{-1}h & 0 & \varepsilon & h \\ 0 & 0 & 0 & 0 & -h^{-1} & 0 \end{pmatrix}. \tag{92}$$

It has the characteristic determinant

$$\rho^6 - (\rho^5 + \rho)(\alpha + \delta + \varepsilon) + (\rho^4 + \rho^2)(3 + \alpha\delta + \alpha\varepsilon + \delta\varepsilon - \gamma g - g^{-1}h) \\ - \rho^3(2\alpha + 2\delta + 2\varepsilon + \alpha\delta\varepsilon - g\gamma\varepsilon - \delta g^{-1}h) + 1. \tag{93}$$

28. We first determine in which cases (92) is conjugate within G with a transformation (55). The conditions that (92) shall replace X and Y, defined by (33), by respectively τY and $-\tau^{-1}X+\sigma Y$, are

$$\begin{aligned}
&\tau A=-b\gamma^{-1}, \quad \tau B=a\gamma+b\delta+c\gamma g, \quad \tau C=b+c\alpha-g^{-1}d+g^{-1}he,\\
&\tau D=cg, \qquad \tau E=c+e\varepsilon-h^{-1}j, \qquad \tau J=eh;\\
&\sigma A-\tau^{-1}a=-B\gamma^{-1}, \quad \sigma B-\tau^{-1}b=A\gamma+B\delta+C\gamma g, \quad \tau D-\tau^{-1}d=Cg,\\
&\tau C-\tau^{-1}c=B+C\alpha-g^{-1}D+g^{-1}hE,\\
&\qquad\qquad\qquad \sigma E-\tau^{-1}e=C+E\varepsilon-h^{-1}J, \quad \sigma J-\tau^{-1}j=hE.
\end{aligned}$$

The first six conditions give

$$\left.\begin{aligned}
&b=-\gamma\tau A, \quad c=g^{-1}\tau D, \quad e=h^{-1}\tau J, \quad j=-h\tau E+g^{-1}h\tau D+\varepsilon\tau J,\\
&a=\gamma^{-1}\tau B+\delta\tau A-\tau D, \quad d=-g\tau C-g\gamma\tau A+\alpha\tau D+\tau J.
\end{aligned}\right\} \tag{94}$$

Substituting these values in the second set of six, we get

$$D=-A(\sigma-\delta), \quad \gamma gC=B(\sigma-\delta), \quad C=E(\sigma-\varepsilon), \quad hD=gJ(\sigma-\varepsilon), \tag{95}$$

$$hE=gC(\sigma-\alpha)-gB=B\{\gamma^{-1}(\sigma-\alpha)(\sigma-\delta)-g\}, \tag{96}$$

$$J=D(\sigma-\alpha)+g\gamma A=A\{g\gamma-(\sigma-\alpha)(\sigma-\delta)\}. \tag{97}$$

Upon setting $\sigma=\rho+\rho^{-1}$, $(93)=0$ becomes

$$W(\sigma)\equiv g\gamma(\sigma-\varepsilon)+g^{-1}h(\sigma-\delta)-(\sigma-\alpha)(\sigma-\delta)(\sigma-\varepsilon)=0, \tag{98}$$

which is therefore the condition that $\rho^2-\sigma\rho+1$ shall be a factor of $\Delta(\rho)$. We assume that this necessary condition for the conjugacy is satisfied.

In view of $(95)_1$ and (97), $(95)_4$ reduces to $AW=0$. Likewise $(95)_3$ reduces to $BW=0$. Hence we may drop $(95)_3$ and $(95)_4$.

In view of relations (94)-(98), Abelian condition (34) becomes*

$$\frac{1}{\tau}=V(A^2+\sigma\gamma^{-1}AB+\gamma^{-2}B^2),$$

$$V\equiv\gamma+h^{-1}g^2\gamma^2+g^{-1}(\sigma-\delta)^2-2h^{-1}g\gamma(\sigma-\alpha)(\sigma-\delta)+h^{-1}(\sigma-\alpha)^2(\sigma-\delta)^2.$$

---

* The only reduction made was in the coefficient of $\gamma^{-1}AB$, viz.,

$$P=\gamma\delta+g^{-1}(2\sigma-\alpha)(\sigma-\delta)^2+h^{-1}\varepsilon g^2\gamma^2-2\varepsilon h^{-1}g\gamma(\sigma-\alpha)(\sigma-\delta)+h^{-1}(\sigma-\alpha)^2(\sigma-\delta)^2,$$

$$\begin{aligned}\therefore\ P-\sigma V=\gamma(\delta-\sigma)+g^{-1}(\sigma-\alpha)(\sigma-\delta)^2+h^{-1}g^2\gamma^2(\varepsilon-\sigma)-2h^{-1}g\gamma(\sigma-\alpha)(\sigma-\delta)(\varepsilon-\sigma)\\
+h^{-1}(\sigma-\alpha)^2(\sigma-\delta)^2(\varepsilon-\sigma).\end{aligned}$$

Eliminating the last term by using (98) multiplied by $h^{-1}(\sigma-\alpha)(\sigma-\delta)$, we find that

$$P-\sigma V=-h^{-1}g\gamma W=0.$$

If $V \neq 0$, the condition merely determines τ, whatever values, not both zero, in F be assigned to A and B. Now $V = 0$ if, and only if, (98) has a double root. Indeed, (98) may be written

$$\sigma - \varepsilon = g^{-1}h(\sigma - \delta)/\{(\sigma - \alpha)(\sigma - \delta) - g\gamma\},$$

since the denominator $\neq 0$. Differentiating* and clearing of fractions, we obtain $V = 0$. If (98) has a double root, not a triple root, the remaining root lies in F and leads to the conjugacy.

THEOREM.—*Within G, transformation* (92) *is conjugate with a type* (55) *if, and only if, its characteristic determinant* (93) *has in F an irreducible factor $\rho^2 - \sigma\rho + 1$ which is not a triple factor.*

29. Consider next (92) when (93) is either irreducible in F or is the product of two irreducible cubic factors (§25). Proceeding as in §26, we find that the only function, aside from a constant factor, which (92) multiplies by a constant κ is

$$X_1 = \xi_1 - \gamma\kappa\eta_1 + g^{-1}D\xi_2 + \kappa^{-1}D\eta_2 + \{\kappa\gamma g + (\kappa + \kappa^{-1} - \alpha)D\}(h^{-1}\xi_3 + \kappa^{-1}\eta_3),$$

where $D \equiv -\kappa^2 + \kappa\delta - 1$, and κ is a root of (93). Let $D' = -\lambda^2 + \lambda\delta - 1$. Then

$$R_{13} = \begin{vmatrix} 1 & -\gamma\kappa \\ 1 & -\gamma\lambda \end{vmatrix} + \begin{vmatrix} g^{-1}D & \kappa^{-1}D \\ g^{-1}D' & g^{-1}D' \end{vmatrix} + \begin{vmatrix} h^{-1}[\kappa\gamma g + (\kappa + \kappa^{-1} - \alpha)D] & \kappa^{-1}[\kappa\gamma g - (\kappa + \kappa^{-1} - \alpha)D] \\ h^{-1}[\lambda\gamma g + (\lambda + \lambda^{-1} - \alpha)D'] & \lambda^{-1}[\lambda\gamma g + (\lambda + \lambda^{-1} - \alpha)D'] \end{vmatrix}$$
$$= (\kappa - \lambda)\{\gamma + g^{-1}(\delta - k)(\delta - l)$$
$$+ h^{-1}[\gamma g - (\alpha - k)(\delta - k)][\gamma g - (\alpha - l)(\delta - l)]\},$$

where $k = \kappa + \kappa^{-1}$, $l = \lambda + \lambda^{-1}$, $m = \mu + \mu^{-1}$ are the roots of (98), viz.,

$$-W(\sigma) \equiv \sigma^3 - \sigma^2(\alpha + \delta + \varepsilon) + \sigma(\alpha\delta + \alpha\varepsilon + \delta\varepsilon - g\gamma - g^{-1}h)$$
$$- \alpha\delta\varepsilon + g\gamma\varepsilon + g^{-1}h\delta = 0. \qquad (98')$$

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* That this argument is valid (here a great simplification) follows since, for $d \neq 0$, the system $\phi(\sigma) = 0$, $\phi'(\sigma) = 0$ is equivalent to

$$\phi/d = 0, \quad (d\phi' - \phi d')/d^2 = 0.$$

Hence $R_{13} = (x - \lambda) R$, where R is symmetrical in k and l, and hence expressible as a function of m:

$$\begin{aligned} R = \gamma - \frac{g^{-1}W(\delta)}{\delta - m} + h^{-1}g^2\gamma^2 + h^{-1}\frac{W(\alpha)\,W(\delta)}{(\alpha - m)(\delta - m)} \\ - h^{-1}\gamma g\,[2\alpha\delta - (\alpha + \delta)(\alpha + \delta + \varepsilon - m) \\ + (\alpha + \delta + \varepsilon)^2 - 2(\alpha\delta + \alpha\varepsilon + \delta\varepsilon - g\gamma - g^{-1}h) - m^2]. \end{aligned}$$

Now $\qquad W(\delta) = g\gamma(\delta - \varepsilon), \quad W(\alpha) = g\gamma(\alpha - \varepsilon) + g^{-1}h(\alpha - \delta).$

Then $h\gamma^{-1}g^{-1}(\alpha - m)(\delta - m)R$ gives a polynomial $m^4 - \ldots.$ But m is a root of the cubic (98′). Multiplying it by m and subtracting from the preceding quartic, we obtain $(\alpha + \delta - \varepsilon)\,W(m)$. Hence $R_{13} = 0$. As in §26, equations (88) hold, while

$$R_{12} = (x - x^{-1})\{\gamma + g^{-1}(\delta - k)^2 + h^{-1}[\gamma g - (\alpha - k)(\delta - k)]^2\},$$

where $k = x + x^{-1}$. Proceeding as in §27, we have the

THEOREM.—*Within the group $SA(6, F)$, two transformations* (92) *are conjugate if they have as common characteristic determinant the product of two irreducible cubic factors. Within $SA(6, p^n)$, two transformations* (92) *are conjugate if they have the same irreducible characteristic determinant.*

30. It remains to consider those transformations of the types (73), (75) and (92) whose characteristic determinants are the cube of an irreducible factor $\rho^2 - \sigma\rho + 1$ in F.

For (75) the conditions are $\delta = \alpha$, $g = -1$ (§23) and then $\Delta(\rho) = (\rho^2 - \alpha\rho + 1)^3$. Denote this (75) by S, and let $x^2 - \alpha x + 1 = 0$. The only functions which S multiplies by x are multiples of $X \equiv x\xi_2 - \xi_3$. If a function Y, independent of X, is replaced by $xY + tX$, then Y is a multiple of

$$\xi_1 - \gamma x\eta_1 + (t - xe)\,\xi_2 + e\xi_3, \quad t(\alpha - 2x) \equiv \gamma x.$$

Here $\alpha - 2x = x^{-1} - x \neq 0$. We take $e = 0$ and set $X_1 = Y$, $X_2 = lX$, where $l = tx^{-1}$. Then S replaces X_1 by $xX_1 + xX_2$. We seek four more functions such that the six shall be conjugate in pairs (the two of a pair being derived from one another by replacing $\varkappa$ by $\varkappa^{-1}$) and such that, when the six are properly ordered, the general Abelian conditions

$$R_{12} = R_{34} = R_{56} \neq 0, \quad R_{ij} = 0 \text{ [unless } (i, j) = (1, 2), (3, 4), (5, 6)]$$

shall hold. One finds the following solution:

$$\begin{aligned} X_1 &= \xi_1 - \gamma\varkappa\eta_1 + l\varkappa\xi_2, \quad X_2 = l(\varkappa\xi_2 - \xi_3), \\ & \qquad X_3 = -\gamma^{-1}\alpha l\xi_1 + 2l\eta_1 + (\alpha - 2\varkappa^{-1})\eta_2 + (\varkappa\alpha - 2)\eta_3, \\ Y_1 &= \xi_1 - \gamma\varkappa^{-1}\eta_1 + \bar{l}\varkappa^{-1}\xi_2, \quad Y_3 = \bar{l}(\varkappa^{-1}\xi_2 - \xi_3), \\ & \qquad Y_2 = -\gamma^{-1}\alpha\bar{l}\xi_1 + 2\bar{l}\eta_1 + (\alpha - 2\varkappa)\eta_2 + (\varkappa^{-1}\alpha - 2)\eta_3, \end{aligned}$$

where, as above, $l = \gamma \div (\alpha - 2\varkappa)$. Here X_1 and Y_1 are conjugate, X_2 and Y_3, X_3 and Y_2. For the transformation from ξ_i, η_i to X_i, Y_i we have every $R_{ij} = 0$ except

$$R_{12} = R_{34} = R_{56} = \gamma(\varkappa - \varkappa^{-1}).$$

The transformation S becomes, in the new variables,

$$\left.\begin{aligned} X_1' &= \varkappa X_1 + \varkappa X_2, \quad X_2' = \varkappa X_2, \quad X_3' = \varkappa X_3 - \varkappa X_1 + \bar{\sigma}X_2 + \bar{\tau}Y_3, \\ Y_1' &= \varkappa^{-1}Y_1 + \varkappa^{-1}Y_3, \quad Y_3' = \varkappa^{-1}Y_3, \quad Y_2' = \varkappa^{-1}Y_2 - \varkappa^{-1}Y_1 + \sigma Y_3 + \tau X_2, \end{aligned}\right\} \quad (100')$$

where

$$\sigma = \frac{\varkappa\alpha - 1 - \alpha^2}{2\varkappa - \alpha} + \gamma^{-1}\beta\varkappa^{-1}(\alpha - 2\varkappa^{-1}), \quad \tau = \frac{\varkappa\alpha - 1 - \alpha^2}{\alpha - 2\varkappa^{-1}} - \gamma^{-1}\beta\varkappa^{-1}(\alpha - 2\varkappa),$$

so that $\varkappa\sigma + \varkappa^{-1}\bar{\sigma} = -1$. Transforming by

$$Y_2' = Y_2 + \lambda X_2, \quad X_3' = X_3 + \bar{\lambda}Y_3, \quad \tau + \lambda(\varkappa - \varkappa^{-1}) = 0,$$

we obtain a like transformation $S_{\sigma, \kappa}$, with $\tau = \bar{\tau} = 0$. Let

$$x_i = X_i / f(\varkappa), \quad y_i = Y_i / f(\varkappa^{-1}), \qquad (i = 1, 2, 3),$$

where $f(\varkappa) = a + b\varkappa$ is not identically zero. Then $S_{\sigma, \kappa}$ retains its form

$$\left.\begin{aligned} x_1' &= \varkappa x_1 + \varkappa x_2, \quad x_2' = \varkappa x_2, \quad x_3' = \varkappa x_3 - \varkappa x_1 + \bar{\sigma}x_2, \\ y_1' &= \varkappa^{-1}y_1 + \varkappa^{-1}y_3, \quad y_3' = \varkappa^{-1}y_3, \quad y_2' = \varkappa^{-1}y_2 - \varkappa^{-1}y_1 + \sigma y_3. \end{aligned}\right\} \quad (100)$$

For the transformation T from ξ_i, η_i to x_i, y_i, we have

$$R_{12} = R_{34} = R_{56} = \frac{\gamma(\varkappa - \varkappa^{-1})}{f(\varkappa)f(\varkappa')}, \quad \text{remaining } R_{ij} = 0.$$

The condition $f(x)f(x^{-1}) \equiv a^2 + aba + b^2 = \gamma$ requires the solution of a quadratic in the field F. We here limit F to the $GF[p^n]$. Then $x^{-1} = x^{p^n}$, and the condition becomes $f^{p^n+1} = \gamma$, all of whose $p^n + 1$ roots f lie in the $GF[p^{2n}]$. Hence (§3, Corollary) for the resulting transformation T,

$$\sum_{i=1}^{3} \begin{vmatrix} \xi_i & \eta_i \\ \xi_i^* & \eta_i^* \end{vmatrix} = \frac{1}{x - x^{-1}} \sum_{i=1}^{3} \begin{vmatrix} x_i & y_i \\ x_i^* & y_i^* \end{vmatrix}, \tag{101}$$

independent of γ. Hence T belongs to $GA(6, p^{2n})$. One consequence is that $S_{\sigma,\kappa}$ belongs to $SA(6, p^{2n})$. Since $x\sigma + x^{-1}\bar{\sigma} = -1$, this may be verified directly.

We proceed to show that $S_{\Sigma,\kappa}$, where $x\Sigma + x^{-1}\bar{\Sigma} = -1$, is conjugate with $S_{\sigma,\kappa}$ within $SA(6, p^{2n})$. Let S be the general transformation (13) of §4. If $S_{\sigma,\kappa}S = SS_{\Sigma,\kappa}$, then

$$S = \begin{pmatrix} \alpha_{11} & 0 & \alpha_{12} & 0 & 0 & 0 \\ 0 & \delta_{11} & 0 & 0 & 0 & \delta_{13} \\ 0 & 0 & \alpha_{11} & 0 & 0 & 0 \\ 0 & \delta_{21} & 0 & \delta_{11} & 0 & \delta_{23} \\ \alpha_{31} & 0 & \alpha_{32} & 0 & \alpha_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_{11} \end{pmatrix}, \tag{102}$$

where

$$\delta_{11}(\Sigma - \sigma) = x^{-1}(\delta_{13} + \delta_{21}), \quad \alpha_{11}(\bar{\Sigma} - \bar{\sigma}) = x(\alpha_{31} + \alpha_{12}). \tag{103}$$

The conjugacy of the variables requires that

$$\delta_{11} = \bar{\alpha}_{11}, \quad \delta_{13} = \bar{\alpha}_{12}, \quad \alpha_{31} = \bar{\delta}_{21}, \quad \alpha_{32} = \bar{\delta}_{23}. \tag{104}$$

The conditions that S shall be special Abelian are then

$$\alpha_{11}\bar{\alpha}_{11} = 1, \quad \delta_{21} = -\alpha_{12}/\alpha_{11}^2, \quad \delta_{21}\bar{\delta}_{21} + \bar{\alpha}_{11}\bar{\delta}_{23} + \alpha_{11}\delta_{23}. \tag{105}$$

The second condition (103) follows from the first, which becomes

$$x\Sigma - x\sigma = \alpha_{11}\bar{\alpha}_{12} - \bar{\alpha}_{11}\alpha_{12}. \tag*{(103)$_1$}$$

For the $GF[p^n]$, this becomes, in view of $(105)_1$,

$$\left(\frac{\alpha_{12}}{\alpha_{11}}\right)^{p^n} - \frac{\alpha_{12}}{\alpha_{11}} = x\Sigma - x\sigma.$$

Raising both members to the power p^n, and applying

$$x\Sigma + x^{-1}\bar{\Sigma} = x\sigma + x^{-1}\bar{\sigma} = -1,$$

we get
$$\left(\frac{\alpha_{12}}{\alpha_{11}}\right)^{p^{2n}} - \left(\frac{\alpha_{12}}{\alpha_{11}}\right)^{p^n} = \varkappa\sigma - \varkappa\Sigma\,.$$

It follows by addition that α_{12}/α_{11} belongs to the $GF[p^{2n}]$, so that in this field $(103)_1$ can be solved for α_{12}/α_{11}. Likewise, $(105)_3$ can be solved for $\alpha_{11}\delta_{23}$ in terms of δ_{21} in the $GF[p^{2n}]$.

THEOREM.—*Within the group $SA(6, p^n)$, two transformations* (75) *having as common characteristic determinant the cube of an irreducible factor $\rho^2 - a\rho + 1$ are conjugate. They are conjugate within $GA(6, p^{2n})$ with the canonical form* (100) *with $\sigma = 1/(\varkappa^{-1} - \varkappa)$.*

31. Consider next transformation (73) when its characteristic determinant (74) is the cube of a factor $\rho^2 - \sigma\rho + 1$ irreducible in F. As in §24,

$$3\sigma = \alpha + \delta, \quad \gamma g + (\delta - \sigma)^2(2\sigma - \alpha) = 0,$$
$$3\sigma^2 - 2\sigma\alpha - 2\sigma\delta + \alpha\delta - 2 - \beta g = 0. \qquad (106)$$

Let $\varkappa^2 - \sigma\varkappa + 1 = 0$. The only function which (73) multiplies by $\varkappa$ is, aside from a constant factor, given by (85). It now becomes

$$X_2 = \xi_1 - \gamma\varkappa\eta_1 + g^{-1}\varkappa^2(\delta - \sigma)\xi_2$$
$$+ (\delta - \sigma)\eta_2 + g^{-1}(\delta - \sigma)(\alpha - \varkappa^{-1})\xi_3 + \varkappa(\delta - \sigma)\eta_3.$$

Replacing $\varkappa$ by $\varkappa^{-1}$, we obtain the conjugate function Y_3. In the present notation, relation (89) gives, in view of $(106)_2$,

$$R_{36} = g^{-1}(\varkappa - \varkappa^{-1})\{\gamma g + (\delta - \sigma)^2(2\sigma - \alpha)\} = 0.$$

We seek a function X_1 such that, under transformation (73),

$$X_1' = \varkappa X_1 + \varkappa X_2.$$

It suffices* to take zero as the coefficient of η_1 in X_1. Then must

$$X_1 = -\xi_1 + g^{-1}(2\varkappa - 2\varkappa^2\sigma + \varkappa^2\delta)\xi_2 + (2\varkappa^{-1} - \delta)\eta_2$$
$$+ g^{-1}\varkappa^{-1}(2\alpha + 2\delta - \sigma - \alpha\delta\varkappa - 2\varkappa^{-1})\xi_3 + (2 - \sigma\varkappa)\eta_3.$$

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* Before attempting the complicated analysis of this section, I first carried out the indicated reduction for the field of order $p^n = 2$. Then must $g = \gamma = \alpha = \beta \equiv 1$, $\delta = 0$. Then (73) is reduced to the canonical form (100), with $\sigma = 1$, by the *special Abelian* transformation

$$X_1' = \xi_1 + \kappa^2\xi_3 + \kappa\eta_3, \qquad X_2' = \xi_1 + \kappa\eta_1 + \kappa^2\xi_2 + \eta_2 + \kappa\xi_3 + \kappa\eta_3,$$
$$Y_1' = \xi_1 + \kappa\xi_3 + \kappa^2\eta_3, \qquad Y_3' = \xi_1 + \kappa^2\eta_1 + \kappa\xi_2 + \eta_2 + \kappa^2\xi_3 + \kappa^2\eta_3,$$
$$Y_2' = \xi_1 + \eta_1 + \kappa\xi_2 + \xi_3 + \eta^3, \quad X_3' = \xi_1 + \eta_1 + \kappa^2\xi_2 + \xi_3 + \eta_3.$$

Denote its conjugate by Y_1. Then $R_{13} = 0$. Let

$$X_3 = q\xi_1 + r\eta_1 + s\xi_2 + v\xi_3 + w\eta_3, \quad Y_2 = \overline{X}_3,$$

it sufficing to take zero as coefficients of η_2. We require that R_{35}, R_{15} and R_{25} shall vanish (i. e., the Abelian conditions connecting X_3 with X_2, X_1, Y_1 respectively):

$$\begin{aligned}
r + \gamma\varkappa q - s(\delta - \sigma) + g^{-1}(\delta - \sigma)(\alpha - \varkappa^{-1})w - \varkappa(\delta - \sigma)v &= 0,\\
-r - s(2\varkappa^{-1} - \delta) + g^{-1}\varkappa^{-1}(2\alpha + 2\delta - \sigma - \alpha\delta\varkappa - 2\varkappa^{-1})w - (2 - \sigma\varkappa)v &= 0,\\
-r - s(2\varkappa - \delta) + g^{-1}\varkappa(2\alpha + 2\delta - \sigma - \alpha\delta\varkappa^{-1} - 2\varkappa)w - (2 - \sigma\varkappa^{-1})v &= 0,
\end{aligned}$$

Subtracting the third from the second and dividing by $\varkappa - \varkappa^{-1}$, we get

$$2s + \sigma v - 3g^{-1}\sigma w = 0. \tag{107}$$

The second and first then give r and q in terms of s, w, v. Forming the function X_3' by which (73) replaces X_3, we obtain for $X_3' - \varkappa X_3 - \varkappa X_1$ [see desired canonical form (100)] the expression

$$\begin{aligned}
(-\varkappa - \varkappa q - r\gamma^{-1})\xi_1 + (q\gamma + r\delta + w\gamma - \varkappa r)\eta_1 + \eta_2(w - \varkappa\delta + 2)&\\
+ [r + s\alpha + v + w\beta - \varkappa s + \varkappa g^{-1}(2\varkappa - 2\varkappa^2\sigma + \varkappa^2\delta)]\xi_2&\\
+ g^{-1}(w\alpha - g\varkappa v + 5\sigma - \alpha\delta\varkappa - 2\varkappa^{-1})\xi_3 + (sg - \varkappa w + 2\varkappa - \varkappa^2\sigma)\eta_3&.
\end{aligned}$$

We seek the conditions under which this shall equal

$$\begin{aligned}
mX_2 + nY_3 \equiv (m + n)\xi_1 - \gamma(m\varkappa + n\varkappa^{-1})\eta_1 + g^{-1}(\delta - \sigma)(m\varkappa^2 + n\varkappa^{-2})\xi_2&\\
+ (\delta - \sigma)(m + n)\eta_2 + g^{-1}(\delta - \sigma)[m(\alpha - \varkappa^{-1}) + n(\alpha - \varkappa)]\xi_3&\\
+ (\delta - \sigma)(\varkappa m + \varkappa^{-1}n)\eta_3&.
\end{aligned}$$

Comparing the coefficients of ξ_1 and η_2, we get

$$(\delta - \sigma)(-\varkappa - \varkappa q - r\gamma^{-1}) = w - \varkappa\delta + 2,$$

Eliminating r and q by the above relations and, by $(106)_2$, also γ, we obtain, after dividing out $(\delta - \sigma)^2$, the condition

$$\phi \equiv -s - v\varkappa + g^{-1}(2\sigma - \varkappa^{-1})w - g^{-1}(\alpha - 2\sigma)(2 - \varkappa\sigma). \tag{108}$$

Comparing, similarly, η_1 and η_3, we get $(\sigma - \delta)(1 - \varkappa\sigma)\phi = 0$. Solving the conditions from η_2 and η_3, we get

$$\left.\begin{aligned}
(\delta - \sigma)(\varkappa - \varkappa^{-1})n &= 2\varkappa w - sg - \varkappa^2\delta + \varkappa^2\sigma,\\
(\delta - \sigma)(\varkappa - \varkappa^{-1})m &= -\sigma w + sg - 2\varkappa^{-1} + 2\varkappa - \varkappa^2\sigma + \delta.
\end{aligned}\right\} \tag{109}$$

Equating the coefficients of ξ_3, we get $\phi = 0$. Equating those of ξ_2, we get $\kappa\phi = 0$. Hence the six conditions reduce to (108) and (109). Adding $2\phi = 0$ to (107) and dividing by $\kappa^{-1} - \kappa$, we get $(110)_1$, by means of which (108) reduces to $(110)_2$:

$$w = gv - 2\kappa(\alpha - 2\sigma), \quad s = \sigma v - 3g^{-1}(\alpha - 2\sigma)\sigma\kappa. \tag{110}$$

In view of these values, we get

$$\left.\begin{aligned} r &= v(\sigma^2 + \sigma\delta - \alpha\delta) - g^{-1}(\alpha - 2\sigma)(4\kappa + 3\sigma\kappa\delta - 2\alpha\delta\kappa), \\ \gamma\kappa q &= g^{-1}(\alpha - 2\sigma)(2\alpha - 4\kappa^{-1} - \sigma\kappa\alpha + \sigma\kappa\delta). \end{aligned}\right\} \tag{111}$$

Now the Abelian function for X_3, Y_3 is

$$\begin{aligned} R_{56} \equiv -\gamma\kappa^{-1}q - r + s(\delta - \sigma) + v\kappa^{-1}(\delta - \sigma) - wg^{-1}(\delta - \sigma)(\alpha - \kappa) \\ = g^{-1}(\alpha - 2\sigma)(\sigma^2 - 4)(\kappa^{-1} - \kappa). \end{aligned}$$

Now R_{34} (on X_2 and Y_2, conjugate to Y_3 and X_3 respectively) equals $-\overline{R}_{56}$, whence $R_{34} = R_{56}$. After some computation, we find that $R_{12} = R_{56}$ and that for the function for Y_2, X_3,

$$R_{45} \equiv \bar{q}r - \bar{r}q + \bar{v}w - \bar{w}v = (\sigma^2 - 4)(v - \bar{v}) - \tau(\kappa^{-1} - \kappa),$$

where

$$\tau \equiv \gamma^{-1}g^{-2}(\alpha - 2\sigma)^2(4 + 3\sigma\delta - 2\alpha\delta)(2\chi\sigma - 4\sigma^2 + 4 - \alpha\sigma^2 + \delta\sigma^2).$$

Since $\sigma^2 - 4 = (\kappa - \kappa^{-1})^2 \neq 0$, and since $\overline{R_{45}} = -R_{45}$, we may choose v to make $R_{45} = 0$. We may, for example, take $(\sigma^2 - 4)v = \tau\kappa^{-1}$.

Making the transformation of variables (99), where now

$$f(\kappa)\, f(\kappa^{-1}) = -g^{-1}(\alpha - 2\sigma)(\sigma^2 - 4),$$

we have for the transformation T of variables from ξ_i, η_i to x_i, y_i,

$$R'_{12} = R'_{34} = R'_{56} = \kappa - \kappa^{-1}, \text{ the remaining } R'_{ij} = 0.$$

Hence (101) holds, so that T belongs to the $GA(6, p^{2n})$. The theorem of §30 therefore holds also for transformations (73).

32. It remains to consider type (92) when its characteristic determinant (93) is the cube of an irreducible factor $\rho^2 - \sigma\rho + 1$.

For the case when F is the $GF[2]$, $\sigma \equiv 1$, $\gamma = g = h = 1$, and

$$\alpha + \delta + \varepsilon \equiv 1, \quad \alpha\delta + \alpha\varepsilon + \delta\varepsilon + 3 \equiv 0, \quad \alpha\delta\varepsilon + \delta + \varepsilon \equiv 1 \pmod 2.$$

By the first, $\alpha(\delta+\varepsilon) \equiv \alpha^2+\alpha \equiv 0$, so that the second gives $\alpha\delta \equiv 1$. Hence $\alpha \equiv \delta \equiv \varepsilon \equiv 1$. Let $\varkappa$ be a root of $\rho^2-\rho+1=0$. A function which (92) mulplies by $\varkappa$ is given by X_1 of §29; it is now written X_2:

$$X_2 = \xi_1 + \varkappa\eta_1 + \varkappa\xi_3 + \eta_3, \quad Y_3 = \overline{X_2} = \xi_1 + \varkappa^2\eta_1 + \varkappa^2\xi_3 + \eta_3.$$

Now $X_1' = \varkappa X_1 + \varkappa X_2$, for $X_1 = \xi_1 + \ldots$ if, and only if,

$$X_1 = \xi_1 + \varkappa\xi_2 + \eta_2 + \eta_3, \quad Y_1 = \overline{X_1} = \xi_1 + \varkappa^2\xi_2 + \eta_2 + \eta_3.$$

Then $R_{12}=1$, $R_{13}=R_{16}=0$. Hence $R_{23}=R_{26}=0$. Let

$$X_3 = q\xi_1 + r\eta_1 + s\xi_2 + t\eta_2 + v\xi_3 + w\eta_3, \quad Y_2 = \overline{X_3}.$$

The Abelian conditions connecting X_3 with X_1, Y_1, X_2, X_3 give

$$r+\varkappa t+s+v=0, \quad r+\varkappa^2 t+s+v=0,$$
$$r+\varkappa q+\varkappa w+v=0, \quad r+\varkappa^2 q+\varkappa^2 w+v=1.$$

Hence $t=0$, $r+s+v=0$, $q+w=1$, $r+v=\varkappa$, so that

$$X_3 = (w+1)\xi_1 + r\eta_1 + \varkappa\xi_2 + (r+\varkappa)\xi_3 + w\eta_3.$$

Let our special (92) replace X_3 by X_3'. We find that

$$X_3' - \varkappa X_3 + \varkappa X_1 \equiv \varkappa^2 X_2 + (r+\varkappa^2+\varkappa w)\,Y_3.$$

Comparing with (100′), we have $\bar{\sigma}=\varkappa^2$, whence $\varkappa\sigma+\varkappa^{-1}\bar{\sigma}=\varkappa^2+\varkappa=-1$. Now $R_{45}=r+\bar{r}+\varkappa\bar{w}+\varkappa^{-1}w$. To make $R_{45}=0$, we dispose of the undetermined r and w by setting $r=w=0$. The resulting transformation from ξ_i, η_i to X_i, Y_i is therefore a *special Abelian* transformation in the $GF[2^2]$, which transforms (92) into (100′) for $\sigma=\tau=\varkappa$.

I refrain from the verification that, for the general $GF[p^n]$, transformation (92) with $\Delta(\rho)=(\rho^2-\sigma\rho+1)^3$ can be transformed into (100) by an Abelian transformation in the $GF[p^{2n}]$ such that relation (101) holds. Indeed, (92) and (100), with $\varkappa^2-\sigma\varkappa+1=0$, are conjugate under *linear* transformation in the $GF[p^{2n}]$, since the characteristic determinant* of each has the single invariant

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* Give it the notation (c_{ij}) and denote the minor of c_{ij} by c_{ij}'. Then, for (100),

$$c_{35}' = (\kappa^{-1}-\rho)^3(\kappa^2+\bar{\sigma}\kappa-\bar{\sigma}\rho), \quad c_{64}' = (\kappa-\rho)^3(\kappa^{-2}+\sigma\kappa^{-1}-\sigma\rho),$$

which have no common factor involving ρ. Likewise, for (92),

$$c_{61}' = \gamma h\rho, \quad c_{41}' = \gamma g(\rho^2-\varepsilon\rho+1).$$

The theorem holds true for (73), since $c_{41}'=\gamma g\rho$, $c_{21}'=\gamma(\rho^4-\rho^3 a-\rho^2 g\beta-\rho a+1)$.

factor $\Delta(\rho)$. Moreover, the conjecture gains weight if we note that types (73), (75), (77) and (92) have led to uniform results thus far, the separation of cases appearing to be not vital but only to avoid the impracticability of treating the parent type (72) subject to numerous lengthy Abelian conditions.

33. The classification in §21 led to four types, (73), (75), (77) and (92). Of these, (77) was reduced to (55) or (75) in §22. In §§23–32, the types (73), (75), (92) were treated separately, the conjugacies which occur in a given type being determined. It remains to determine the conjugacies occurring between the various transformations of the aggregate of the three types. This, however, requires no further computation, at least when we take F to be the $GF[p^n]$. Either a transformation of the aggregate is conjugate within $SA(6, p^n)$ with a type (55) or else it can be transformed into one of the canonical forms (91) and (100) by a general Abelian transformation T in the $GF[p^{nt}]$, where t is the common degree of the irreducible factors of $\Delta(\rho)$, the multiplier (μ of §2) of T being unity or $x - x^{-1}$, where $\Delta(x) = 0$.

THEOREM.—*Within* $SA(6, p^n)$, *two transformations, selected from the aggregate of the types* (73), (75), (77), (92), *but such that neither is conjugate with* (55), *are conjugate with each other, if, and only if, they have the same characteristic determinant.*

Types left for treatment by induction, §§34–40.

34. It remains to consider (16), (20), (23) and (55). Of these, (20) is the product of a special Abelian transformation on $\xi_1, \eta_1, \xi_2, \eta_2$ by one on ξ_i, η_i $(i>2)$, occurring only when $m \geqq 3$. Each of the other three is a product of a special Abelian transformation on ξ_1, η_1 by one on ξ_i, η_i $(i > 1)$.

35. Let first $m = 2$. Then (16) is the product of $T_{1,\kappa}T$, where T is a binary transformation of determinant unity on ξ_2, η_2. We apply to T the methods of §1. Let F be the $GF[p^n]$. The resulting types are $T_{1,\kappa}T_{2,\lambda}$, $T_{1,\kappa}T_{2,\pm1}L_{2,\mu}$, $\mu = 1$ or a particular not-square ν, and $T_{1,\kappa}S_1$. We may replace S_1 by the ultimate canonical form $T_{2,\lambda}$, $\lambda^{p^n+1} = 1$, λ in the $GF[p^{2n}]$.

For (23) and (55) we similarly reduce not only the partial transformation on ξ_2, η_2, but also that on ξ_1, η_1.

Among the types arising for an arbitrary field F is $L_{1,\mu}L_{2,\tau}$, μ and τ in F. We proceed to investigate their conjugacy within G. Now

$$L_{1,\mu}L_{2,\tau}, S = SL_{1,\rho}L_{2,\sigma},$$

where S is a general quaternary transformation, if and only if,

$$\rho\beta_{11}=\rho\beta_{12}=0,\quad \sigma\beta_{21}=\sigma\beta_{22}=0,\quad \rho\delta_{11}=\mu\alpha_{11},\quad \rho\delta_{12}=\tau\alpha_{12},$$
$$\mu\beta_{11}=\mu\beta_{21}=0,\quad \tau\beta_{12}=\tau\beta_{22}=0,\quad \sigma\delta_{22}=\tau\alpha_{22},\quad \sigma\delta_{21}=\mu\alpha_{21}.$$

If $\tau=0$, $\mu\neq 0$, $\sigma\neq 0$, then β_{11}, β_{21}, β_{22}, δ_{22} are zero and

$$\rho\beta_{12}=0,\quad \rho\delta_{12}=0,\quad \sigma\delta_{21}=\mu\alpha_{21}.$$

Then $\alpha_{21}\delta_{21}=1$, by an Abelian condition, whence $\sigma/\mu=\alpha_{21}^2$. Also $\rho=0$, the determinant of S not vanishing. Likewise if $\tau=\sigma=0$, $\mu\neq 0$, $\rho\neq 0$, then $\beta_{11}=\beta_{12}=\delta_{12}=0$, $\alpha_{11}\delta_{11}=1$, $\rho\delta_{11}=\mu\alpha_{11}$, so that $\rho/\mu=\alpha_{11}^2$. *Hence the only transformations* $L_{1,\rho}L_{2,\sigma}$ *conjugate with* $L_{1,\mu}$ *are* $L_{1,\mu\alpha^2}$ *and* $L_{2,\mu\alpha^2}$.

Lastly, if μ, τ, ρ, σ are all $\neq 0$,

$$S=\begin{pmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ 0 & \rho^{-1}\mu\alpha_{11} & 0 & \rho^{-1}\tau\alpha_{12} \\ \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ 0 & \sigma^{-1}\mu\alpha_{21} & 0 & \sigma^{-1}\tau\alpha_{22} \end{pmatrix}.$$

The Abelian conditions reduce to the following

$$\mu\alpha_{11}^2+\tau\alpha_{12}^2=\rho,\quad \mu\alpha_{21}^2+\tau\alpha_{22}^2=\sigma,\quad \mu\alpha_{11}\alpha_{21}+\tau\alpha_{12}\alpha_{22}=0, \tag{112}$$

$$\alpha_{11}\gamma_{21}-\alpha_{21}\gamma_{11}+\alpha_{12}\gamma_{22}-\alpha_{22}\gamma_{12}=0. \tag{113}$$

If the α_{ij} can be chosen in F to satisfy (112), then the γ_{ij} can be chosen to satisfy (113). It is possible to take $\alpha_{11}=0$ if and only if ρ/τ and σ/μ are squares in F. Let next $\alpha_{11}\neq 0$, so that $\alpha_{22}\neq 0$, and set $\alpha_{12}=\varkappa\alpha_{11}$, $\alpha_{21}=\lambda\alpha_{22}$. Conditions (112) then become

$$\alpha_{11}^2(\mu+\tau\varkappa^2)=\rho,\quad \alpha_{22}^2(\tau+\mu\lambda^2)=\sigma,\quad \mu\lambda+\tau\varkappa=0. \tag{112'}$$

Multiplying the first by τ and the second by μ, and applying the third, we see that $\tau\rho/\sigma\mu$ must be a square in F. Let this condition be satisfied, so that the second condition (112′) will determine α_{22} in F when the first determines α_{11} in F. Consider, therefore, the first and third conditions only. If μ/ρ is a square, we may take $\varkappa=\lambda=0$. Let next μ/ρ be a not-square in F. If $\varkappa$ can be determined in F such that

$$\mu/\rho+\varkappa^2\tau/\rho=\text{square}, \tag{114}$$

then α_{11} and λ can be determined in F to satisfy the first and third conditions (112′). If τ/ρ is a square, $\tau\rho$ is a square, so that $\sigma\mu$ is a square by hypothesis and consequently also μ/σ; but in this case we found that solutions were given by taking $\alpha_{11}=0$.

If μ, τ, ρ, σ *are each not zero,* $L_{1,\mu}L_{2,\tau}$ *and* $L_{1,\rho}L_{2,\sigma}$ *are conjugate within* G *except in the cases* (i), $\tau\rho/\sigma\mu$ *is a not-square in* F; (ii), $\tau\rho/\sigma\mu$ *is a square, while* μ/ρ, τ/ρ *and* $\mu/\rho+\kappa^2\tau/\rho$ *are not-squares for every* κ *in* F.

For the $GF[p^n]$, $p>2$, $\mu/\rho+\kappa^2\tau/\rho$ can be made a square (or a not-square) by choice of κ, so that the single type of the form $L_{1,\;i}L_{2,\nu_j}$ is conjugate with $L_{1,1}L_{2,1}$. For the field of all real numbers, the not-squares are negative so that for case (ii) $\mu/\rho+\kappa^2\tau/\rho$ is a not-square for every κ; hence no type $L_{1,\nu_i}L_{2,\nu_j}$ is conjugate with $L_{1,1}L_{2,1}$ or with $L_{1,1}L_{2,\nu_k}$.

36. Summary of all canonical forms for $m=2$ and F the $GF[p^n]$. Let $\mu=1$ or a particular not-square ν if $p>2$, $\mu=1$ if $p=2$. The canonical forms with all elements in the $GF[p^n]$ are

$$T_{1,\kappa}T_{2,\lambda};\quad T_{1,\kappa}T_{2,\pm1}L_{2,\mu};\quad L_{1,1}L_{2,\mu}T_{1,\pm1}T_{2,\pm1};\quad (19)\text{ with }\kappa\neq\kappa^{-1};$$

$$\text{additional for } p>2\text{: } L_{1,1}T_{1,-1}L_{2,\mu};\quad L_{1,\nu}T_{1,-1}L_{2,\mu};\quad A_\mu T_{1,\pm1}T_{2,\pm1}=(25);$$

$$\text{additional for } p=2\text{: } R_\beta=(26);\quad R_{1,2,1}=(19)\text{ for }\kappa=1.$$

The canonical forms with not all elements in the $GF[p^n]$ are

(63) with $\kappa^{p^n+1}=1$, $\kappa^2\neq1$; (68) with κ, λ in the $GF[p^{2n}]$ or $GF[p^{4n}]$;
$T_{1,\kappa}T_{2,\lambda}$ with $\kappa^{p^n-1}=1$, $\lambda^{p^n+1}=1$, $\lambda^2\neq1$; $T_{1,\kappa}T_{2,\lambda}$ with $\kappa^{p^n+1}=\lambda^{p^n+1}$, $\kappa^2\neq1$, $\lambda^2\neq1$; $L_{1,\mu}T_{1,\pm1}T_{2,\lambda}$ with $\lambda^{p^n+1}=1$, $\lambda^2\neq1$.

From this list we obtain a list of canonical forms no two of which are conjugate by retaining but one of the set

$$T_{1,a^{\pm1}}T_{2,b},\quad T_{1,a^{\pm1}}T_{2,b^{-1}},\quad T_{1,b}T_{2,a^{\pm1}},\quad T_{1,b^{-1}}T_{2,a^{\pm1}}.$$

37. Consider next (16) for $m=3$ and F the $GF[p^n]$. Changing the notation, we write it $T_{3,\omega}S$, where ω is a mark $\neq0$ of F such that $\omega\neq\omega^{-1}$, and S is a special Abelian substitution on ξ_1, η_1, ξ_2, η_2. We give to S in turn the canonical forms of §36, except (19), which is treated in §38. However each form is subdivided as far as convenient in determining all the transformations of G commutative with $T_{3,\omega}S$. We therefore follow the order (but with the present notation) of the expanded list in Transactions, vol. 2 (1901), p. 132.

Type $T_{1,\kappa}T_{2,\lambda}T_{3,\omega}$, κ, κ^{-1}, λ, λ^{-1}, ω, ω^{-1} all different.

Transforming by a suitable product of the M_i and P_{ij}, we can bring any one of the 6 multipliers into the first place and hence its inverse into the second place, any one of the remaining 4 into the third place, etc. Hence there are

$$\tfrac{1}{48}(p^n-3)(p^n-5)(p^n-7) \text{ or } \tfrac{1}{48}(2^n-2)(2^n-4)(2^n-6)$$

non-conjugate forms of this type, according as $p>2$ or $p=2$. A form is commutative only with the $(p^n-1)^3$ transformations $T_{1,a}T_{2,b}T_{3,c}$.

Type $T_{1,\kappa}T_{2,\kappa^{-1}}T_{3,\omega}$, κ, κ^{-1}, ω, ω^{-1} all distinct. There are

$$\tfrac{1}{4}(p^n-3)(p^n-5) \text{ or } \tfrac{1}{4}(2^n-2)(2^n-4)$$

non-conjugate forms of this type. Each is commutative only with $AT_{3,\tau}$, where A is the second matrix on p. 108 of Transactions. The number of commutative transformations is

$$(p^{2n}-1)(p^{2n}-p^n)(p^n-1).$$

Type $T_{1,\omega}T_{2,\omega}T_{3,\omega}$, $\omega\neq\omega^{-1}$. The number of non-conjugates is

$$\tfrac{1}{2}(p^n-3) \text{ or } \tfrac{1}{2}(2^n-2).$$

By a general theorem (Linear Groups, pp. 229–233), which is used repeatedly here to avoid computation, the type is commutative only with

$$\begin{pmatrix} \alpha_{11} & 0 & \alpha_{12} & 0 & \alpha_{13} & 0 \\ 0 & \delta_{11} & 0 & \delta_{12} & 0 & \delta_{13} \\ \alpha_{21} & 0 & \alpha_{22} & 0 & \alpha_{23} & 0 \\ 0 & \delta_{21} & 0 & \delta_{22} & 0 & \delta_{23} \\ \alpha_{31} & 0 & \alpha_{32} & 0 & \alpha_{33} & 0 \\ 0 & \delta_{31} & 0 & \delta_{32} & 0 & \delta_{33} \end{pmatrix}. \qquad (115)$$

Its determinant equals $|\alpha_{ij}|.|\delta_{ij}|$. The Abelian conditions are

$$\alpha_{i1}\delta_{i1}+\alpha_{i2}\delta_{i2}+\alpha_{i3}\delta_{i3}=1, \quad \alpha_{i1}\delta_{j1}+\alpha_{i2}\delta_{j2}+\alpha_{i3}\delta_{j3}=0,$$
$$(i,j=1,2,3;\ i\neq j).$$

Then $|\alpha_{ij}|.|\delta_{ij}|=1$. Hence the δ_{ij} are uniquely determined in terms of the α_{ij}. The group of the commutative substitutions is therefore simply isomorphic with the ternary general linear homogeneous group (Linear Groups, p. 77) of order

$$(p^{3n}-1)(p^{3n}-p^n)(p^{3n}-p^{2n}).$$

Type $T_{1,\kappa}T_{2,\pm1}T_{3,\omega}$, κ, κ^{-1}, ω, ω^{-1} all distinct. There are

$$\tfrac{1}{4}(p^n-3)(p^n-5) \text{ or } \tfrac{1}{8}(2^n-2)(2^n-4)$$

non-conjugate forms. Each is commutative only with

$$\xi_1' = a\xi_1, \quad \eta_1' = a^{-1}\eta_1, \quad \xi_2' = e\xi_2 + f\eta_2, \quad \eta_2' = g\xi_2 + h\eta_2,$$
$$\xi_3' = b\xi_3, \quad \eta_2' = b^{-1}\eta_3, \qquad (116)$$

where $eh - fg = 1$. There are $(p^n - 1)^2(p^{2n} - 1)p^n$ of these.

Type $T_{1,\omega^{-1}}T_{2,\pm 1}T_{3,\omega}$, $\omega \neq \omega^{-1}$. The number of non-conjugates is

$$p^n - 3 \text{ or } \tfrac{1}{2}(2^n - 2).$$

Each is commutative only with AB, A as above and B any binary transformation of determinant 1 on ξ_2, η_2, giving

$$(p^{2n} - 1)(p^{2n} - p^n) \,.\, (p^{2n} - 1)p^n.$$

Type $T_{1,\varkappa}T_{2,\pm 1}L'_{2,\mu}T_{3,\omega}$, $\varkappa$, $\varkappa^{-1}$, ω, ω^{-1} all distinct, μ as in §36. The number of non-conjugate forms is

$$\tfrac{1}{2}(p^n - 3)(p^n - 5) \text{ or } \tfrac{1}{8}(2^n - 2)(2^n - 4).$$

Each is commutative only with (116) with $f = 0$, $e = h$, giving

$$2p^n(p^n - 1)^2 \text{ or } 2^n(2^n - 1)^2.$$

Type $T_{1,\omega^{-1}}T_{2,\pm 1}L'_{2,\mu}T_{3,\omega}$. The number of non-conjugates is

$$2(p^n - 3) \text{ or } \tfrac{1}{2}(2^n - 2).$$

Each is commutative only with $AT_{2,\pm 1}L'_{2,g}$, A as above, giving

$$2p^n.(p^{2n} - 1)(p^{2n} - p^n) \text{ or } 2^n.(2^{2n} - 1)(2^{2n} - 2^n).$$

Type $T_{1,\pm 1}T_{2,\pm 1}T_{3,\omega}$, $\omega \neq \omega^{-1}$. The number of non-conjugates is

$$p^n - 3 \text{ or } \tfrac{1}{2}(2^n - 2).$$

Each is commutative with $p^{4n}(p^{4n} - 1)(p^{2n} - 1)(p^n - 1)$ transformations.

Type $L_{1,\mu}T_{1,\pm 1}T_{2,\pm 1}T_{3,\omega}$. The number of non-conjugates is

$$2(p^n - 3) \text{ or } \tfrac{1}{2}(2^n - 2).$$

By Trans., bottom of p. 114, each is commutative with only

$$2p^{4n}(p^{2n} - 1)\,.\,(p^n - 1) \text{ or } 2^{4n}(2^{2n} - 1)\,.\,(2^n - 1).$$

Type $L_{1,1}L_{2,1}T_{1,\pm 1}T_{2,\pm 1}T_{3,\omega}$. The number of non-conjugates is

$$p^n - 3 \text{ or } \tfrac{1}{2}(2^n - 2).$$

If $\varepsilon = \pm 1$ according as $p^n = 4l \pm 1$, there are

$$2p^{3n}(p^n - \varepsilon) \,.\, (p^n - 1) \text{ or } 2^{4n} \,.\, (2^n - 1)$$

transformations commutative with each (Trans., p. 115).

Type $L_{1,1} L_{2,\nu} T_{1,\pm 1} T_{2,\pm 1} T_{3,\omega}$, for $p > 2$. There are $p^n - 3$ non-conjugate types, each commutative with only

$$2p^{3n}(p^n + \varepsilon) \,.\, (p^n - 1).$$

Type $R_{1,2,1} T_{3,\omega}$, $p = 2$. Each of the $\frac{1}{2}(2^n - 2)$ non-conjugate forms is commutative with $2^{4n}(2^{2n} - 1) \,.\, (2^n - 1)$ transformations (Tr., p. 111).

Type $R_\beta T_{3,\omega}$, $p = 2$. Each of the $2^n - 2$ non-conjugate forms is commutative with only $2 \,.\, 2^{2n} \,.\, (2^n - 1)$ transformations (Tr., p. 112).

Type $A_\mu T_{1,\pm 1} T_{2,\pm 1} T_{3,\omega}$, $p > 2$. Each of the $2(p^n - 3)$ non-conjugate forms is commutative with $2p^{2n} \,.\, (p^n - 1)$ transformations (Tr., p. 113).

Type $T_{1,a} T_{2,\lambda} T_{3,\omega}$, a, a^{-1}, ω, ω^{-1}, all distinct marks of the $GF[p^n]$, $\lambda^{p^n+1} = 1$, $\lambda^2 \neq 1$. The number of non-conjugate forms is

$$\tfrac{1}{16}(p^n - 1)(p^n - 3)(p^n - 5) \text{ or } \tfrac{1}{16} 2^n (2^n - 2)(2^n - 4).$$

Each is commutative with the $(p^n - 1)^2(p^n + 1)$ $T_{1,c} T_{2,\lambda'} T_{3,d}$.

Type $T_{1,\omega} T_{2,\lambda} T_{3,\omega}$, ω and λ as before. There are

$$\tfrac{1}{4}(p^n - 1)(p^n - 3) \text{ or } \tfrac{1}{4} 2^n (2^n - 2)$$

non-conjugate forms, each commutative with only

$$(p^{2n} - 1)(p^{2n} - p^n) \,.\, (p^n + 1).$$

Type $T_{1,\pm 1} T_{2,\lambda} T_{3,\omega}$, ω and λ as before. There are

$$\tfrac{1}{2}(p^n - 1)(p^n - 3) \text{ or } \tfrac{1}{4} 2^n (2^n - 2)$$

non-conjugate forms, each commutative with

$$p^n (p^{2n} - 1)(p^n + 1) \,.\, (p^n - 1).$$

Type $T_{1,\pm 1} L_{1,\mu} T_{2,\lambda} T_{3,\omega}$, μ, λ, ω as before. There are

$$(p^n - 1)(p^n - 3) \text{ or } \tfrac{1}{4} 2^n (2^n - 2)$$

non-conjugate forms, each commutative (Tr., p. 119) with

$$2p^n (p^n + 1) \,.\, (p^n - 1) \text{ or } 2^n (2^n + 1) \,.\, (2^n - 1).$$

Type $T_{1,\sigma}T_{2,\sigma^{p^n}}T_{3,\omega}$, $\sigma^{p^{2n}+1}=1$, $\sigma^2\neq 1$. There are

$$\tfrac{1}{8}(p^{2n}-1)(p^n-3) \text{ or } \tfrac{1}{8}2^{2n}(2^n-2)$$

non-conjugate types (Tr., §18), each commutative with

$$(p^{2n}+1)(p^n-1).$$

Type $T_{1,\sigma}T_{2,\sigma^{p^n}}T_{3,\omega}$, $\sigma^{p^{2n}-1}=1$, $\sigma^{p^n\pm 1}\neq 1$. There are

$$\tfrac{1}{8}(p^n-1)^2.(p^n-3) \text{ or } \tfrac{1}{8}2^n(2^n-2).(2^n-2)$$

non-conjugate types (Tr., §19), each commutative with

$$(p^{2n}-1)(p^n-1).$$

Type $T_{1,\kappa}T_{2,\lambda}T_{3,\omega}$, $\varkappa^{p^n+1}=\lambda^{p^n+1}=1$, $\varkappa^2\neq 1$, $\lambda^2\neq 1$, $\lambda\neq\varkappa$, $\varkappa^{-1}$. There are

$$\tfrac{1}{16}(p^n-1)(p^n-3).(p^n-3) \text{ or } \tfrac{1}{16}2^n(2^n-2).(2^n-2)$$

non-conjugate forms (Tr., §20), each commutative with

$$(p^n+1)^2(p^n-1).$$

Type $T_{1,\kappa}T_{2,\kappa}T_{3,\omega}$, $\varkappa^{p^n+1}=1$, $\varkappa^2\neq 1$. There are

$$\tfrac{1}{4}(p^n-1)(p^n-3) \text{ or } \tfrac{1}{4}2^n(2^n-2)$$

non-conjugate forms (Tr., §21), each commutative with

$$(p^n+1)p^n(p^{2n}-1).(p^n-1).$$

Type (63) $T_{3,\omega}$, $\varkappa^{p^n+1}=1$, $\varkappa^2\neq 1$. There are

$$\tfrac{1}{4}(p^n-1)(p^n-3) \text{ or } \tfrac{1}{4}2^n(2^n-2)$$

non-conjugate forms (Tr., §22), each commutative with

$$p^n(p^n+1).(p^n-1).$$

Type $T_{1,-1}T_{3,\omega}$, $p>2$. Each of the $\tfrac{1}{2}(p^n-3)$ non-conjugate forms is commutative (Tr., p. 109) with $[p^n(p^{2n}-1)]^2(p^n-1)$.

Types $L_{1,\mu}T_{1,-1}T_{3,\omega}$ and $L_{2,\mu}T_{1,-1}T_{3,\omega}$, $p>2$. Each of the $2(p^n-3)$ non-conjugate forms is commutative with (Tr., §9)

$$2p^{2n}(p^{2n}-1).(p^n-1).$$

Types $L_{1,1}T_{1,-1}L_{2,\mu}T_{3,\omega}$ and $L_{1,\nu}T_{1,-1}L_{2,\mu}T_{3,\omega}$, $p>2$. Each of the $2(p^n-3)$ non-conjugate forms is commutative with only $4p^{2n}(p^n-1)$ transformations (Tr., §10).

38. We pass to transformation (20) for $m = 3$ and F the $GF[p^n]$. We give the binary transformation of determinant unity on ξ_3, η_3 one of the canonical form of §1.

Type (19) $T_{3,\alpha}$, $x, x^{-1}, \alpha, \alpha^{-1}$ all distinct marks of the $GF[p^n]$. Note that P_{12} transforms (19) into a similar transformation with x^{-1} in place of x. Hence there are

$$\tfrac{1}{4}(p^n - 3)(p^n - 5) \text{ or } \tfrac{1}{4}(2^n - 2)(2^n - 4)$$

non-conjugate forms of our type. Each is commutative only with

$$\begin{array}{c|c} \Sigma & O \\ \hline O & \begin{array}{cc} e & o \\ o & e^{-1} \end{array} \end{array}, \quad \Sigma \equiv \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & ba^{-2} & 0 \\ 0 & 0 & a^{-1} & 0 \\ b & 0 & 0 & a \end{pmatrix}, \tag{117}$$

where O denotes a matrix all of whose elements are zero. The number of transformations is thus $p^n(p^n - 1)^2$.

Type (19) $T_{3,\kappa}$. The number of non-conjugates is

$$\tfrac{1}{2}(p^n - 3) \text{ or } \tfrac{1}{2}(2^n - 2).$$

Each is commutative only with the linear transformations*

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & b_1 & 0 & 0 & c_1 \\ 0 & 0 & a_1 & 0 & 0 & 0 \\ b & 0 & 0 & a & c & 0 \\ d & 0 & 0 & 0 & e & 0 \\ 0 & 0 & d_1 & 0 & 0 & e_1 \end{pmatrix}. \tag{118}$$

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* This result may be verified directly or derived from the general theory, first rearranging the variables. Then the first transformation below is commutative only with the second :

$$\begin{array}{c|cccccc} & \xi_1 & \eta_2 & \xi_3 & \xi_2 & \eta_1 & \eta_3 \\ \hline \xi_1' = & \kappa & 0 & 0 & 0 & 0 & 0 \\ \eta_2' = & \kappa & \kappa & 0 & 0 & 0 & 0 \\ \xi_3' = & 0 & 0 & \kappa & 0 & 0 & 0 \\ \xi_2' = & 0 & 0 & 0 & \kappa^{-1} & 0 & 0 \\ \eta_1' = & 0 & 0 & 0 & \kappa^{-1} & \kappa^{-1} & 0 \\ \eta_3' = & 0 & 0 & 0 & 0 & 0 & \kappa^{-1} \end{array} \qquad \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ b & a & c & 0 & 0 & 0 \\ d & 0 & e & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & b_1 & a_1 & c_1 \\ 0 & 0 & 0 & d_1 & 0 & e_1 \end{pmatrix}.$$

The Abelian conditions are

$$\left.\begin{array}{lll} aa_1 = 1, & ee_1 = 1, & -a_1 b + b_1 a - c_1 c = 0, \\ -a_1 d - ec_1 = 0, & & -ad_1 + ce_1 = 0, \end{array}\right\} \qquad (118')$$

whence

$$a_1 = \frac{1}{a}, \quad e_1 = \frac{1}{e}, \quad d_1 = \frac{c}{ae}, \quad c_1 = \frac{-d}{ae}, \quad b_1 = \frac{b}{a^2} - \frac{cd}{a^2 e}.$$

Hence c, b, d, a, e are arbitrary, with $a \neq 0$, $e \neq 0$, so that there are exactly $p^{3n}(p^n - 1)^2$ commutative transformations.

Type (19) $T_{3, \pm 1}$. Each of the $p^n - 3$ or $\frac{1}{2}(2^n - 2)$ non-conjugate forms is commutative only with the transformations

$$\begin{array}{c|c} \Sigma & O \\ \hline O & \begin{matrix} r & s \\ t & u \end{matrix} \end{array}, \quad ru - ts = 1, \qquad (119)$$

Σ as in (117). There are $p^n(p^n - 1) . p^n(p^{2n} - 1)$ of these.

Type (19) $T_{3, \pm 1} L'_{3, \mu}$. The number of non-conjugates is

$$2(p^n - 3) \text{ or } \tfrac{1}{2}(2^n - 2).$$

Each is commutative only with (119) for $s = 0$, $r = u$, giving

$$2p^n . p^n(p^n - 1) \text{ or } 2^n . 2^n(2^n - 1).$$

Type (19) $T_{3, \lambda}$, $\lambda^{p^n + 1} = 1$, $\lambda^2 \neq 1$. There are

$$\tfrac{1}{4}(p^n - 1)(p^n - 3) \text{ or } \tfrac{1}{4} 2^n (2^n - 2)$$

non-conjugate forms. Each is commutative only with (117) for $e^{p^n + 1} = 1$, giving $p^n(p^n - 1)(p^n + 1)$ transformations.

39. Consider next transformation (23) for $m = 3$ and F the $GF[p^n]$. Changing the notation of the variables, we have $WT_{3, \pm 1} L'_{3, \tau}$ where W is a special Abelian transformation on $\xi_1, \eta_1, \xi_2, \eta_2$. Let the characteristic determinant of W be $D(\rho)$. The case when $D(\rho)$ has in F a root $\varkappa \neq \varkappa^{-1}$ was excluded by hypothesis (§8). We first suppose that every root of $D(\rho)$ is a root of $x^2 = 1$. Then W may be taken to be $R_{1, 2, 1,} R_\beta$, $A_\mu T_{1, \pm 1} T_{2, \pm 1}$ or a product of certain $L_{i, a}$,

$T_{i,-1}$, $i=1, 2$. In the last case, we replace $L'_{3,\tau}$ by its conjugate $L_{3,\tau}$ and have the following preliminary types:*

$$\begin{gathered} I;\ T_{3,-1};\ L_{1,\mu};\ L_{1,\mu}T_{1,-1};\ L_{2,\mu}T_{1,-1};\ L_{1,\mu}L_{2,1};\ L_{1,\mu}T_{1,-1}L_{2,\lambda}; \\ L_{1,\mu}T_{1,-1}L_{2,1}T_{2,-1};\ L_{1,\lambda}T_{1,-1}L_{2,1}L_{3,\mu};\ L_{1,1}L_{2,1}L_{3,\mu}; \\ (\lambda=\mu=1 \text{ if } p=2;\ \lambda,\ \mu=1,\ \nu \text{ if } p>2), \end{gathered} \tag{120}$$

together with their products by $T \equiv T_{1,-1}T_{2,-1}T_{3,-1}$.

For $p>2$, $L_{1,1}L_{2,1}L_{3,1}$ is not conjugate with† $L_{1,1}L_{2,1}L_{3,\nu}$ since

$$L_{1,1}L_{2,1}L_{3,1}S = SL_{1,\nu}L_{2,\nu}L_{3,\nu}$$

requires that, when S is given the form (13),

$$\beta_{ij}=0, \quad \alpha_{ij}=\nu\delta_{ij} \quad (i, j=1, 2, 3).$$

The Abelian conditions on S then reduce to

$$\delta_{1i}^2+\delta_{2i}^2+\delta_{3i}^2=1/\nu, \quad \delta_{1i}\delta_{1j}+\delta_{2i}\delta_{2j}+\delta_{3i}\delta_{3j}=0 \quad (i, j=1, 2, 3;\ i \neq j),$$

together with linear homogeneous conditions on the γ_{ij}. The conditions on the δ_{ij} are precisely the conditions that the ternary transformation (δ_{ij}) shall replace $\xi_1^2+\xi_2^2+\xi_3^2$ by $\frac{1}{\nu}(\xi_1^2+\xi_2^2+\xi_4^2)$. But such a replacement is impossible‡ if ν is a not-square.

Next, $L_{1,1}$ and $L_{1,\nu}$ are not conjugate within G. For the conditions for $L_{1,1}(13)=(13)L_{1,\nu}$ are

$$\beta_{11}=\beta_{12}=\beta_{21}=\beta_{31}=\beta_{13}=\alpha_{21}=\alpha_{31}=\delta_{12}=\delta_{13}=0, \quad \alpha_{11}=\nu\delta_{11}$$

Then an Abelian condition gives $\alpha_{11}\delta_{11}=1$, whence $\nu=\alpha_{11}^2$.

That the two forms of the type $L_{1,\mu}L_{2,1}$ are not conjugate within G follows from the enumeration below of the commutative transformations.

That no two of the three types $L_{1,\mu}$, $L_{1,a}L_{2,b}$, $L_{1,c}L_{2,d}L_{3,e}$ are conjugate follows from the theory of canonical forms. Also, by the invariance of the roots of the characteristic equation, two conjugate forms (120) must contain the same number of factors $T_{i,-1}$, if $p>2$.

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* Noting that, by §35, $L_{i,1}L_{j,1}$ and $L_{i,\nu}L_{j,\nu}$ are conjugate. † Ib.

‡ Jordan, Traité, p. 171, for the case of modulus p and $m=3$. I have a direct proof of the following theorem for an arbitrary field F. No linear transformation with coefficients in F multiplies $\mu_1\xi_1^2+\mu_2\xi_2^2+\ldots+\mu_m\xi_m^2$ by a not-square in F, when m is odd.

If, for $p > 2$, S transforms $L_{1,\lambda} T_{1,-1} L_{2,1} L_{3,1}$ into $L_{1,\lambda} T_{1,-1} L_{2,1} L_{3,\nu}$, whose p^{th} powers equal $T_{1,-1}$, then S is the product of a special Abelian transformation on ξ_1, η_1 by one on ξ_2, η_2, ξ_3, η_3. The latter would transform $L_{2,1} L_{3,1}$ into $L_{2,1} L_{3,\nu}$, contrary to §33. Similarly, if S transforms $L_{1,\mu} T_{1,-1} L_{2,1}$ into $L_{1,\lambda} T_{1,-1} L_{2,\nu}$, it would transform $L_{2,1}$ into $L_{2,\nu}$.

Treating the remaining cases similarly, we find that no two of the transformations (120) are conjugate within G.

Type $L_{1,1} L_{2,1} L_{3,\mu} T$. It is commutative with (13) only if

$$\begin{gathered}\delta_{11} = \alpha_{11}, \quad \delta_{12} = \alpha_{12}, \quad \delta_{13} = \mu\alpha_{13}, \quad \delta_{21} = \alpha_{21}, \quad \beta_{ij} = 0, \qquad (i, j = 1, 2, 3), \\ \delta_{22} = \alpha_{22}, \quad \delta_{33} = \alpha_{33}, \quad \delta_{23} = \mu\alpha_{23}, \quad \alpha_{31} = \mu\delta_{31}, \quad \alpha_{32} = \mu\delta_{32}.\end{gathered}$$

The Abelian conditions on (13) then reduce to

$$\left.\begin{array}{l}\alpha_{11}^2 + \alpha_{12}^2 + \mu\alpha_{13}^2 = 1, \quad \alpha_{21}^2 + \alpha_{22}^2 + \mu\alpha_{23}^2 = 1, \quad \alpha_{31}^2 + \alpha_{32}^2 + \mu\alpha_{33}^2 = \mu; \\ \alpha_{i1}\alpha_{j1} + \alpha_{i2}\alpha_{j2} + \mu\alpha_{i3}\alpha_{j3} = 0, \qquad (i, j = 1, 2, 3;\ i < j);\end{array}\right\} \tag{121}$$

$$\alpha_{i1}\gamma_{j1} - \alpha_{j1}\gamma_{i1} + \alpha_{i2}\gamma_{j2} - \alpha_{j2}\gamma_{i2} + \alpha_{i3}\gamma_{j3} - \alpha_{j3}\gamma_{i3} = 0, \quad (i, j = 1, 2, 3;\ i < j). \tag{122}$$

Now, if $p > 2$, conditions (121) are precisely the conditions* for the invariance of $\xi_1^2 + \xi_2^2 + \frac{1}{\mu}\xi_3^2$ under the ternary transformation (α_{ij}), so that the number of sets of solutions is $2(p^{2n} - 1)p^n$. For $p = 2$, then $\mu = 1$ and the first conditions (121) give $\alpha_{i3} = 1 + \alpha_{i1} + \alpha_{i2}$ for $i = 1, 2, 3$. Then the second set of three become

$$A_{11}A_{22} + A_{12}A_{21} = 1, \quad A_{11}A_{32} + A_{12}A_{31} = 1, \quad A_{22}A_{31} + A_{21}A_{32} = 1,$$

where $A_{ij} \equiv 1 + \alpha_{ij}$. The first two of these may be solved for A_{11} and A_{12}, since their determinant is unity by the third:

$$A_{11} = A_{21} + A_{31}, \quad A_{12} = A_{22} + A_{32}.$$

The third has $2^n(2^{2n} - 1)$ sets of solutions. This is therefore the number of sets of solutions α_{ij} of (121). The determinant of (13) equals $|\alpha_{ij}|^2$. Hence $|\alpha_{ij}| = \pm 1$, as is evident from what precedes if $p > 2$. Now if all the determinants of order 3 in the 3 by 9 matrix of the coefficients of the γ_{ij} in (122) all vanish, we find that the minor of each α_{i3} in $|\alpha_{ij}|$ vanishes, whereas the latter

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* **American Journal, Vol. XXI (1899), p. 195, formulae (1′), (2′) for $s = m = 3$.**

$\neq 0$. Hence relations (122) determine three of the γ_{ij} in terms of the remaining ones and the α_{ij}. The number of transformations commutative with $L_{1,1}L_{2,1}L_{3,\mu}$ is therefore

$$2(p^{2n}-1)p^n . p^{6n} \text{ or } (2^{2n}-1)\,2^n . 2^{6n}.$$

Types $L_{1,\mu}T$, $L_{1,\mu}L_{2,1}T$. The conditions for $L_{1,\mu}L_{2,\tau}(13)=(13)L_{1,\mu}L_{2,\tau}$ are

$$\beta_{11}=\beta_{12}=\beta_{21}=\beta_{31}=\beta_{13}=\delta_{13}=\alpha_{31}=0, \quad \tau\beta_{22}=0, \quad \tau\beta_{32}=0,$$
$$\tau\beta_{23}=0, \quad \tau\alpha_{32}=0,$$
$$\tau\delta_{23}=0, \quad \delta_{11}=\alpha_{11}, \quad \tau\delta_{21}=\mu\alpha_{21}, \quad \mu\delta_{12}=\tau\alpha_{12}, \quad \tau\delta_{22}=\tau\alpha_{22}.$$

According as $\tau=0$ or $\tau=1$, we obtain

$$\begin{pmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} & \alpha_{13} & \gamma_{13} \\ 0 & \alpha_{11} & 0 & 0 & 0 & 0 \\ 0 & \gamma_{21} & \alpha_{22} & \gamma_{22} & \alpha_{23} & \gamma_{23} \\ 0 & \delta_{21} & \beta_{22} & \delta_{22} & \beta_{23} & \delta_{23} \\ 0 & \gamma_{31} & \alpha_{32} & \gamma_{32} & \alpha_{33} & \gamma_{33} \\ 0 & \delta_{31} & \beta_{32} & \delta_{32} & \beta_{33} & \delta_{33} \end{pmatrix}, \quad \begin{pmatrix} \alpha_{11} & \gamma_{11} & \mu\delta_{12} & \gamma_{12} & \alpha_{13} & \gamma_{13} \\ 0 & \alpha_{11} & 0 & \delta_{12} & 0 & 0 \\ \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} & \alpha_{23} & \gamma_{23} \\ 0 & \mu\alpha_{21} & 0 & \alpha_{22} & 0 & 0 \\ 0 & \gamma_{31} & 0 & \gamma_{32} & \alpha_{33} & \gamma_{33} \\ 0 & \delta_{31} & 0 & \delta_{32} & \beta_{33} & \delta_{33} \end{pmatrix}.$$

For the first, $\alpha_{11}=\pm 1$ and $\gamma_{21}, \delta_{21}, \gamma_{31}, \delta_{31}$ are determined by the Abelian conditions, while the matrix obtained by deleting the first two rows and columns is that of a quaternary special Abelian transformation. The number of transformations commutative with $L_{1,\mu}$ is therefore

$$2p^{5n}.p^{4n}(p^{4n}-1)(p^{2n}-1) \text{ or } 2^{5n}.\,2^{4n}(2^{4n}-1)(2^{2n}-1).$$

For the second the Abelian conditions are

$$\alpha_{11}^2+\mu\delta_{12}^2=1, \quad \alpha_{11}\alpha_{21}+\delta_{12}\alpha_{22}=0, \quad \mu\alpha_{21}^2+\alpha_{22}^2=1, \tag{123}$$
$$\alpha_{33}\delta_{33}-\beta_{33}\gamma_{33}=1, \quad \alpha_{11}\gamma_{21}-\gamma_{11}\alpha_{21}+\mu\delta_{12}\gamma_{22}-\gamma_{12}\alpha_{22}+\alpha_{13}\gamma_{23}-\gamma_{13}\alpha_{23}=0, \tag{124}$$
$$\alpha_{11}\gamma_{31}+\mu\delta_{12}\gamma_{32}+\alpha_{13}\gamma_{33}-\gamma_{13}\alpha_{33}=0, \quad \alpha_{11}\delta_{31}+\mu\delta_{12}\delta_{32}+\alpha_{13}\delta_{33}-\gamma_{13}\beta_{33}=0, \tag{125}$$
$$\alpha_{21}\gamma_{31}+\alpha_{22}\gamma_{32}+\alpha_{23}\gamma_{33}-\gamma_{23}\alpha_{33}=0, \quad \alpha_{21}\delta_{31}+\alpha_{22}\delta_{32}+\alpha_{23}\delta_{33}-\gamma_{23}\beta_{33}=0. \tag{126}$$

From (123), $\alpha_{21}^2=\delta_{12}^2$, $\alpha_{22}=\pm\alpha_{11}$, $\alpha_{11}(\alpha_{21}\pm\delta_{12})=0$. The number N of sets of solutions of (123) is

$$2(p^n+\epsilon) \text{ if } \mu=\nu\,; \quad 2(p^n-\epsilon) \text{ if } \mu=1, p>2\,; \quad 2^n \text{ if } p=2$$

where $\epsilon=\pm 1$ according as $p^n=4l\pm 1$.

In view of (124), we can solve (125) for α_{13} and γ_{13}, and (126) for α_{23} and γ_{23}. Conceive of these values being substituted in $(124)_2$. The coefficients of γ_{21} and γ_{11} remain α_{11} and $-\alpha_{21}$ respectively, not both of which are zero. Hence either γ_{11} or γ_{21} is determined. Hence for each of the $p^n(p^{2n}-1)N$ sets of solutions of (123) and $(124)_1$, for the 8 coefficients entering, the 12 remaining coefficients are connected by relations which determine 5 of them. Hence $L_{1,\mu}L_{2,1}$ is commutative with exactly

$$2(p^n+\varepsilon)p^n(p^{2n}-1)p^{7n} \text{ if } \mu=\nu,\ p>2;$$

$$2(p^n-\varepsilon)p^n(p^{2n}-1)p^{7n} \text{ if } \mu=1,\ p>2$$

$$2^n.\,2^n(2^{2n}-1)\,2^{7n} \text{ if } p=2.$$

Type $L_{1,\mu}T_{1,-1}T$, $p>2$. A transformation commutative with it is commutative with its p^{th} power $T_{1,-1}T$ and hence is the product of a special Abelian transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on ξ_1, η_1 by one on ξ_2, η_2, ξ_3, η_3. Being commutative with $L_{1,\mu}$, $a=d=\pm 1$, $c=0$. Hence there are

$$2\,p^n.\,p^{4n}(p^{4n}-1)(p^{2n}-1).$$

Type $L_{2,\mu}T_{1,-1}T$, $p>2$. A substitution commutative with it must be commutative with both $T_{1,-1}$ and $L_{2,\mu}$. By Trans., bottom of page 114, or by type $L_{1,\mu}$, their number is

$$2p^{4n}(p^{2n}-1)\,.\,p^n(p^{2n}-1).$$

Type $L_{1,\mu}T_{1,-1}L_{2,\lambda}T$, $p>2$, μ, $\lambda=1,\nu$. A transformation commutative with it is commutative with $T_{1,-1}$ and $L_{1,\mu}L_{2,\lambda}$. The number is thus

$$2p^{4n}(p^{2n}-1)\,.\,2p^n.$$

Type $L_{2,\mu}T_{2,-1}L_{3,1}T_{3,-1}T$, $p>2$. Its p^{th} power is $T_{2,-1}T_{3,-1}T$. A commutative transformation is, therefore, the product of type (17), Trans., p. 115, on ξ_2, η_2, ξ_3, η_3 by a transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant unity on ξ_1, η_1. The number is thus

$$2p^{3n}(p^n+\varepsilon)\,.\,p^n(p^{2n}-1) \text{ if } \mu=\nu;\quad 2p^{3n}(p^n-\varepsilon)\,p^n(p^{2n}-1) \text{ if } \mu=1.$$

Type $L_{1,\lambda}T_{1,-1}L_{2,\mu}L_{3,1}T$, $p>2$. A commutative transformation is of the form in the preceding type with $c=0$, $a=d=\pm 1$:

$$2p^{3n}(p^n+\epsilon)\,.\,2p^n \text{ if } \mu=\nu\,;\quad 2p^{3n}(p^n-\epsilon)\,.\,2p^n \text{ if } \mu=1\,.$$

Type $T_{3,-1}T$, $p>2$. The number of commutative ones is

$$p^n(p^{2n}-1)\,.\,p^{4n}(p^{4n}-1)(p^{2n}-1).$$

Type $R_{1,2,1}L'_{3,1}$, $p=2$. For the sequence ξ_1, η_2, ξ_2, η_1, ξ_3, η_3 of the variables, the commutative transformation (as given by inspection by the general theory) is the first one below. For the initial sequence ξ_1, η_1, ξ_2, η_2, ξ_3, η_3, we have the second:

$$\begin{pmatrix} a & 0 & e & 0 & \alpha & 0 \\ b & a & f & e & \beta & \alpha \\ g & 0 & c & 0 & \gamma & 0 \\ h & g & d & c & \delta & \gamma \\ \varkappa & 0 & \rho & 0 & r & 0 \\ \tau & \varkappa & \sigma & \rho & s & r \end{pmatrix}, \quad \begin{pmatrix} a & 0 & e & 0 & \alpha & 0 \\ h & c & d & g & \delta & \gamma \\ g & 0 & c & 0 & \gamma & 0 \\ b & e & f & a & \beta & \alpha \\ \varkappa & 0 & \rho & 0 & r & 0 \\ \tau & \rho & \sigma & \varkappa & s & r \end{pmatrix}.$$

The Abelian conditions on the second give

$$\alpha=0,\quad \gamma=0,\quad r=1,\quad a\rho+e\varkappa=0,\quad c\varkappa+g\rho=0, \tag{127}$$

$$he+bc+da+fg=0,\quad h\rho+c\tau+d\varkappa+g\sigma+\delta r=0, \tag{128}$$

$$ac+eg=1,\quad b\rho+e\tau+f\varkappa+a\sigma+\beta r=0. \tag{129}$$

From the last two in (127), $\rho=\varkappa=0$, since the determinant is not zero by $(129)_1$. Now $(128)_2$ and $(129)_2$ determine δ and β. Since e and c are not both zero, $(128)_1$ determines either h or b. Hence of the 18 coefficients, 8 are uniquely determined in terms of a, c, e, g [connected by $(129)_1$], d, f, τ, σ, s and either h or b. The number of transformations is therefore

$$2^n(2^{2n}-1)\,2^{6n}.$$

This result suggests that the type is perhaps conjugate with $L_{1,1}L_{2,1}L_{3,1}$. That this is in fact the case follows since the Abelian transformation

$$\xi'_1=\eta_2+\eta_3,\quad \eta'_1=\xi_1+\xi_3,\quad \xi'_2=\eta_1+\eta_3,$$
$$\eta'_2=\xi_2+\xi_3,\quad \xi'_3=\xi_1+\xi_2+\xi_3,\quad \eta'_3=\eta_1+\eta_2+\eta_3$$

transforms $L_{1,1}L_{2,1}L_{3,1}$ into $R_{1,2,1}L_{3,1}$.

Type $R_{1,2,1}$ for $p=2$. The auxiliary and final are now

$$\begin{pmatrix} a & 0 & e & 0 & 0 & 0 \\ b & a & f & e & \alpha & \gamma \\ g & 0 & c & 0 & 0 & 0 \\ h & g & d & c & \beta & \delta \\ \varkappa & 0 & \rho & 0 & r & s \\ \tau & 0 & \sigma & 0 & t & u \end{pmatrix}, \quad \begin{pmatrix} a & 0 & e & 0 & 0 & 0 \\ h & c & d & g & \beta & \delta \\ g & 0 & c & 0 & 0 & 0 \\ b & e & f & a & \alpha & \gamma \\ \varkappa & 0 & \rho & 0 & r & s \\ \tau & 0 & \sigma & 0 & t & u \end{pmatrix}.$$

The Abelian conditions on the second are

$$ac+eg=1, \quad e\varkappa+a\rho+\alpha s+\gamma r=0, \quad e\tau+a\sigma+\alpha u+\gamma t=0, \tag{130}$$

$$ru+ts=1, \quad he+bc+da+fg+\beta\gamma+\alpha\delta=0, \tag{131}$$

$$c\varkappa+g\rho+\beta s+r\delta=0, \quad c\tau+g\sigma+\beta u+t\delta=0. \tag{132}$$

By the last two in (130), α and γ are determined, since the determinant $\neq 0$ by $(131)_1$. Likewise β and δ are determined by (132). Then $(131)_2$ determines either h or b, neither occurring in the other conditions. There are $[2^n(2^{2n}-1)]^2$ sets, a, c, e, g, r, u, t, s, satisfying $(130)_1$, $(131)_1$. Then $d, f, \varkappa, \rho, \tau, \sigma$, and either h or b, are arbitrary. The number of transformations is thus $2^{9n}(2^{2n}-1)^2$.

Type $R_\beta L'_{3,\tau}$, $p=2$. Denote it by $V_{\beta,\tau}$. If S be defined by (13), we have $V_{\beta,\tau}S=SV_{B,T}$ if, and only if,

$$\begin{pmatrix} \alpha_{11} & 0 & 0 & 0 & 0 & 0 \\ \beta_{11} & \alpha_{11} & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \\ \delta_{12} & 0 & \alpha_{11} & 0 & 0 & 0 \\ \beta_{21} & 0 & \delta_{12} & \alpha_{11} & \tau\delta_{13} & 0 \\ \alpha_{31} & 0 & 0 & 0 & \alpha_{33} & \gamma_{33} \\ \beta_{31} & 0 & T\alpha_{31} & 0 & \beta_{33} & \delta_{33} \end{pmatrix},$$

$$\tau\gamma_{33}=T\gamma_{33}=0, \quad \tau\delta_{33}=T\alpha_{33}, \quad \beta\alpha_{11}+\beta_{12}+\delta_{12}=B\alpha_{11}+\beta_{21}.$$

The Abelian conditions all reduce to

$$\alpha_{11}=1, \quad \beta_{21}+\delta_{12}^2+\beta_{12}+\tau\delta_{13}^2=0, \quad \beta_{31}+T\alpha_{31}\delta_{12}+\beta_{13}\delta_{33}+\delta_{13}\beta_{33}=0,$$
$$\alpha_{31}+\beta_{13}\gamma_{33}+\delta_{13}\alpha_{33}=0, \quad T\alpha_{31}+\tau\delta_{13}\delta_{33}=0, \quad \alpha_{33}\delta_{33}-\beta_{33}\gamma_{33}=1.$$

If $\tau=1$, then $\gamma_{33}=0$, $\delta_{33}=T\alpha_{33}$, $\alpha_{33}\delta_{33}=1$, whence $T\neq 0$. By an earlier transformation, we can make $T=1$. If $\tau=0$, then $T\gamma_{33}=T\alpha_{33}=0$, whence $T=0$. Hence, if the two are conjugate, we may set $T=\tau=1$ or 0.

For $T = \tau = 1$, then $\gamma_{33} = 0$, $\alpha_{33} = \delta_{33} = 1$, $\alpha_{11} = 1$, $\alpha_{31} = \delta_{13}$,

$$\beta_{31} = \alpha_{31}\delta_{12} + \beta_{13} + \delta_{13}\beta_{33}, \quad \beta_{21} = \beta_{12} + \delta_{12}^2 + \delta_{13}^2, \quad \delta_{13}^2 = \delta_{12} + \delta_{12}^2 + \beta + B.$$

Hence $V_{\beta,1}$ is transformed into $V_{B,1}$ by the resulting Abelian transformation in which δ_{12}, β_{12}, β_{33}, β_{13}, β_{11} are arbitrary* and the others determined. It follows that the number commutative with $V_{\beta,1}$ is 2^{5n}.

For $T = \tau = 0$, we must have $\delta_{12}^2 + \delta_{12} = \beta + B$, so that a necessary condition is $f(\beta) = f(B)$, f defined by (28). Let this condition be satisfied, so that two values of δ_{12} can be found in the $GF[2^n]$. Then for any set of solutions of $\alpha_{33}\delta_{33} - \beta_{33}\gamma_{33} = 1$ and β_{11}, β_{12}, β_{13}, δ_{13} arbitrary, the conditions merely determine β_{11}, β_{21}, α_{31}, β_{31}. Hence there are exactly two non-conjugate forms $V_{0,0}$ and $V_{b,0}$, b a particular solution of $f(\eta) = 1$. It follows also that there are exactly $2 \,.\, 2^n (2^{2n} - 1)\, 2_{4n}$ Abelian transformations commutative with R_β, viz.,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \beta_{11} & 1 & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \\ \delta_{12} & 0 & 1 & 0 & 0 & 0 \\ \beta_{12} + \delta_{12} & 0 & \delta_{12} & 1 & 0 & 0 \\ \beta_{13}\gamma_{33} + \delta_{13}\alpha_{33} & 0 & 0 & 0 & \alpha_{33} & \gamma_{33} \\ \beta_{13}\delta_{33} + \delta_{13}\beta_{33} & 0 & 0 & 0 & \beta_{33} & \delta_{33} \end{pmatrix}, \qquad \begin{array}{l} \delta_{12} = 0 \text{ or } 1, \\ \alpha_{33}\delta_{33} - \beta_{33}\gamma_{33} = 1. \end{array}$$

Types $A_\mu T_{1,\pm 1} T_{2,\pm 1} T_{3,c} L'_{3,\tau}$, $c^2 = 1$, $\tau = 0, 1, \nu$, $p > 2$. It suffices to consider $A_\mu L'_{3,\tau}$, $A_\mu T_{3,-1} L'_{3,\tau}$ and their products by $T_{1,-1} T_{2,-1} T_{3,-1}$.

Type $A_\mu T_{3,-1} L'_{3,\tau} T \equiv Y_{\mu,\tau}$, $p > 2$. The power p^2 of it gives $T_{3,-1} T$, since A_μ is of period p if $p > 3$ and of period 9 if $p = 3$ (Trans., p. 113). Hence if S transforms $Y_{\mu,\tau}$ into $Y_{M,t}$, S is commutative with $T_{3,-1}$ and hence is the product of a special Abelian transformation Y on ξ_1, η_1, ξ_2, η_2 by one on ξ_3, η_3. The latter, say $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, must transform $L'_{3,\tau}$ into $L'_{3,t}$. Hence

$$b\tau = 0, \quad bt = 0, \quad d\tau = at, \quad ad - bc = 1.$$

If $\tau = 0$, then $t = 0$. If $\tau \neq 0$, then $ad = 1$, $d\tau = at$, whence t/τ must be a square. Applying earlier transformations, we may take $\tau = t = 1$ or ν. Since Y shall transform A_μ into A_M, where μ, $M = 1$, ν, it follows from §9 that $\mu = M$. Hence there are 6 non-conjugate forms $Y_{\mu,\tau}$ ($\mu = 1, \nu$; $\tau = 0, 1, \nu$). We may now readily determine the transformations S commutative with each $Y_{\mu,\tau}$. By §9, for $\alpha=\alpha'=\mu$, we have

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* Taking these zero, whence $\beta_{31} = 0$, $\beta_{21} = \delta_{13}^2$, $\delta_{13}^2 = \beta + B$, I verified the conjugacy.

$$Y=\begin{pmatrix} \alpha_{11} & 0 & 0 & 0 \\ \beta_{11} & \alpha_{11} & \beta_{12} & \delta_{12} \\ -\delta_{12} & 0 & \alpha_{11} & 0 \\ -\beta_{12}-\mu\delta_{12} & 0 & \mu\delta_{12} & \alpha_{11} \end{pmatrix}, \quad \begin{array}{l} \alpha_{11}^2=1, \\ 2\alpha_{11}\beta_{12}+\mu\alpha_{11}\delta_{12}-\mu\delta_{12}^2=0, \end{array}$$

the number of which is $2p^{2n}$. Further, we have

$$b\tau=0, \quad d\tau=a\tau, \quad ad-bc=1,$$

the number of solutions of which is $p^n(p^{2n}-1)$ or $2p^n$ according as $\tau=0$ or $\tau\neq 0$. In the respective cases, $Y_{\mu,\tau}$ is commutative with

$$2p^{3n}(p^{2n}-1) \text{ or } 4p^{3n}.$$

Type $A_\mu L'_{3,\tau}T\equiv X_{\mu,\tau}$, $\mu=1,\nu$; $\tau=0,1,\nu$, $p>2$. The conditions for $X_{\mu,\tau}S=SX_{M,t}$ require that

$$S=\begin{pmatrix} \alpha_{11} & 0 & 0 & 0 & 0 & 0 \\ \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} & \beta_{13} & \delta_{13} \\ \alpha_{21} & 0 & \alpha_{11} & 0 & 0 & 0 \\ -\beta_{12}-\mu\delta_{12} & 0 & \mu\delta_{12} & \delta_{11} & \tau\delta_{13} & 0 \\ \alpha_{31} & 0 & 0 & 0 & \alpha_{33} & \gamma_{33} \\ \beta_{31} & 0 & -t\alpha_{31} & 0 & \beta_{33} & \delta_{33} \end{pmatrix}, \quad \begin{array}{rl} \mu\delta_{11}= & M\alpha_{11}, \\ \mu\delta_{12}= & -M\alpha_{21}, \\ \tau\gamma_{33}= & 0, \\ t\gamma_{33}= & 0, \\ \tau\delta_{33}= & t\alpha_{33}. \end{array}$$

The Abelian conditions all give

$$\alpha_{11}\delta_{11}=1, \quad \delta_{11}\alpha_{21}+\alpha_{11}\delta_{12}=0, \quad 2\delta_{11}\beta_{12}+\mu\delta_{11}\delta_{12}-\mu\delta_{12}^2-\tau\delta_{13}^2=0,$$
$$t\delta_{11}\alpha_{31}+\tau\delta_{13}\delta_{33}=0, \quad -\delta_{11}\beta_{31}+t\delta_{12}\alpha_{31}+\beta_{13}\delta_{33}-\delta_{13}\beta_{33}=0,$$
$$\alpha_{33}\delta_{33}-\beta_{33}\gamma_{33}=1, \quad -\delta_{11}\alpha_{31}+\beta_{13}\gamma_{33}-\delta_{13}\alpha_{33}=0.$$

Then $\mu\delta_{11}^2=M$, whence $M=\mu=1$ or ν; $\delta_{11}=\alpha_{11}=\pm 1$, $\alpha_{21}=-\delta_{12}$. If $\tau=0$, then $t=0$. If $\tau\neq 0$, then $\gamma_{33}=0$, $\alpha_{33}\delta_{33}=1$, $t=\tau\delta_{33}^2$, whence $t=\tau$. Hence no two of the six forms $X_{\mu,\tau}$ are conjugate.

Setting $M=\mu$, $t=\tau$, we get $\delta_{11}=\alpha_{11}=\pm 1$, $\alpha_{21}=-\delta_{12}$ and

$$\tau\gamma_{33}=0, \quad \tau\delta_{33}=\tau\alpha_{33}, \quad \pm 2\beta_{12}\pm\mu\delta_{12}-\mu\delta_{12}^2-\tau\delta_{13}^2=0, \tag{133}$$
$$\pm\tau\alpha_{31}+\tau\delta_{13}\delta_{33}=0, \quad \pm\beta_{31}=\tau\delta_{12}\alpha_{31}+\beta_{13}\delta_{33}-\delta_{13}\beta_{33}, \tag{134}$$
$$\alpha_{33}\delta_{33}-\beta_{33}\gamma_{33}=1, \quad \pm\alpha_{31}=\beta_{13}\gamma_{33}-\delta_{13}\alpha_{33}. \tag{135}$$

If $\tau\neq 0$, then $\gamma_{33}=0$, $\delta_{33}=\alpha_{33}$, $\alpha_{33}^2=1$, $\alpha_{31}=\mp\delta_{13}\alpha_{33}$, while the last conditions in (133) and (134) determine β_{12} and β_{31}. Since β_{11}, δ_{12}, β_{13}, δ_{13}, β_{33} are arbitrary, we obtain $4p^{5n}$ transformations commutative with $X_{\mu,\tau}$, $\tau \neq 0$. If $\tau = 0$, β_{11}, δ_{12}, β_{13} and δ_{13} are arbitrary, while α_{11}, δ_{11}, α_{21}, β_{12}, β_{31}, α_{31} are determined. Hence there are $2p^{4n}.\, p^n(p^{2n} - 1)$ transformations commutative with A_μ.

It remains to take as W, in $WT_{3,\pm 1}L_{3,\tau}$, a quaternary canonical form for which $D(\rho)$ has four roots not in the $GF[p^n]$, or two roots ± 1 and two roots not in the $GF[p^n]$. Consider the subdivisions necessary for the types $T_{1,c}L_{1,b}T_{2,\lambda}T_{3,\pm 1}L_{3,\tau}$, where (to the end of this section) $\lambda^{p^n+1} = 1$, $\lambda^2 \neq 1$. A transformation which transforms one such type into another is the product of $T_{2,\delta}$, $\delta^{p^n+1} = 1$, by a special Abelian transformation A on $\xi_1, \eta_1, \xi_3, \eta_3$. Hence they must have the same c and the same sign $\pm$ in view of the invariance of the roots of their characteristic equations. If $c = \pm 1$, we may restrict $L_{1,b}L_{3,\tau}$, to the forms I, $L_{1,\mu}$, $L_{1,1}L_{3,\mu}$. If $c = \mp 1$, $p > 2$, it suffices to take as $T_{1,c}L_{1,b}T_{3,\pm 1}L_{3,\tau}$ one of the non-conjugate forms $T_{1,-1}$, $T_{1,-1}L_{1,\mu}$, $T_{1,-1}L_{3,\mu}$, $T_{1,-1}L_{1,\mu}L_{3,\tau}$, where $\tau = 1$ or ν.

Type $T_{1,\pm 1}T_{2,\lambda}T_{3,\pm 1}$. The number of non-conjugate forms is

$$p^n - 1 \text{ or } \tfrac{1}{2}\, 2^n.$$

The number of Abelian transformations C commutative with any one of the forms is

$$p^{4n}(p^{4n} - 1)(p^{2n} - 1)(p^n + 1).$$

Type $T_{1,\pm 1}L_{1,\mu}T_{2,\lambda}T_{3,\pm 1}$. The number of non-conjugates is

$$2(p^n - 1) \text{ or } \tfrac{1}{2}\, 2^n.$$

The number of the C is (Trans., bottom of p. 114)

$$2p^{4n}(p^{2n} - 1)(p^n + 1) \text{ or } 2^{4n}(2^{2n} - 1)(2^n + 1).$$

Type $T_{1,\pm 1}L_{1,1}T_{2,\lambda}T_{3,\pm 1}L_{3,\mu}$. There are

$$2(p^n - 1) \text{ or } \tfrac{1}{2}\, 2^n$$

non-conjugate forms. The number of the C is (Tr., p. 115)

$2p^{3n}(p^n + \varepsilon)(p^n + 1)$ if $\mu = \nu$, $p > 2$;

$2p^{3n}(p^n - \varepsilon)(p^n + 1)$ if $\mu = 1$, $p > 2$; $2^{4n}(2^n + 1)$ if $p = 2$.

Type $T_{1,-1}T_{2,\lambda}$, $p > 2$. There are $\frac{1}{2}(p^n - 1)$ non-conjugate forms, each commutative with $p^{2n}(p^{2n} - 1)^2(p^n + 1)$ transformations.

Types $T_{1,-1}L_{1,\mu}T_{2,\lambda}$, $T_{1,-1}T_{2,\lambda}L_{3,\mu}$, $p > 2$. The total number of non-conjugates is $2(p^n-1)$. Each is commutative (Tr., §9) with

$$2p^{2n}(p^{2n}-1)(p^n+1).$$

Type $T_{1,-1}L_{1,\mu}T_{2,\lambda}L_{3,\tau}$, $p>2$. Each of the $2(p^n-1)$ non-conjugate forms is commutative (Tr., §10) with $4p^{2n}(p^n+1)$.

Type $T_{1,\sigma}T_{2,\sigma^{p^n}}T_{3,\pm 1}L_{3,\mu}$, $\sigma^{p^{2n}+1}=1$, $\sigma^2\neq 1$. Each of the

$$p^{2n}-1 \text{ or } \tfrac{1}{4}\,2^{2n}$$

non-conjugate forms is commutative with $2p^n(p^{2n}+1)$ or $2^n(2^{2n}+1)$.

Type $T_{1,\sigma}T_{2,\sigma^{p^n}}T_{3,\pm 1}$, σ as before. Each of the

$$\tfrac{1}{2}(p^{2n}-1) \text{ or } \tfrac{1}{4}\,2^{2n}$$

non-conjugate forms is commutative with $p^n(p^{2n}-1)(p^{2n}+1)$.

Type $T_{1,\sigma}T_{2,\sigma^{p^n}}T_{3,\pm 1}L_{3,\mu}$, $\sigma^{p^{2n}-1}=1$, $\sigma^{p^n\pm 1}\neq 1$. Each of the

$$(p^n-1)^2 \text{ or } \tfrac{1}{4}\,2^n(2^n-2)$$

non-conjugate forms is commutative with $2p^n(p^{2n}-1)$ or $2^n(2^{2n}-1)$.

Type $T_{1,\sigma}T_{2,\sigma^{p^n}}T_{3,\pm 1}$, σ as in preceding. Each of the

$$\tfrac{1}{2}(p^n-1)^2 \text{ or } \tfrac{1}{4}\,2^n(2^n-2)$$

non-conjugate forms is commutative with $p^n(p^{2n}-1)^2$.

Type $T_{1,\varkappa}T_{2,\lambda}T_{3,\pm 1}L_{3,\mu}$, $\varkappa^{p^n+1}=1$, $\lambda^{p^n+1}=1$, λ, λ^{-1}, $\varkappa$, $\varkappa^{-1}$ all distinct. The number of non-conjugate forms is

$$\tfrac{1}{2}(p^n-1)(p^n-3) \text{ or } \tfrac{1}{8}\,2^n(2^n-2).$$

Each is commutative with $2p^n(p^n+1)^2$ or $2^n(2^n+1)^2$.

Type $T_{1,\varkappa}T_{2,\lambda}T_{3,\pm 1}$, $\varkappa$, λ as before. There are

$$\tfrac{1}{4}(p^n-1)(p^n-3) \text{ or } \tfrac{1}{8}\,2^n(2^n-2)$$

non-conjugate forms each commutative with $p^n(p^{2n}-1)(p^n+1)^2$.

Type $T_{1,\varkappa}T_{2,\varkappa}T_{3,\pm 1}L_{3,\mu}$, $\varkappa^{p^n+1}=1$, $\varkappa^2\neq 1$. There are

$$2(p^n-1) \text{ or } \tfrac{1}{2}\,2^n$$

non-conjugate forms, each commutative with (Tr., §21)

$$2p^n.\,(p^n+1)\,p^n(p^{2n}-1) \text{ or } 2^n.\,(2^n+1)\,2^n(2^{2n}-1).$$

Type $T_{1,\varkappa}T_{2,\varkappa}T_{3,\pm 1}$, $\varkappa$ as before. There are p^n-1 or $\frac{1}{2}\,2^n$ non-conjugate forms, each commutative with

$$p^n(p^{2n}-1)\,.\,(p^n+1)\,p^n(p^{2n}-1).$$

Type (63) $T_{3,\pm 1}L_{3,\mu}$, $x^{p^n+1}=1$, $x^2\neq 1$. There are

$$2(p^n-1) \text{ or } \tfrac{1}{2}\,2^n$$

non-conjugate forms, each commutative with (Tr., §22)

$$2p^n.\,p^n(p^n+1) \text{ or } 2^n.\,2^n(2^n+1).$$

Type (63) $T_{3,\pm 1}$, x as before. Each of the p^n-1 or $\frac{1}{2}\,2^n$ non-conjugate forms is commutative with $p^n(p^n+1)\,.\,p^n(p^{2n}-1)$.

40. Finally, consider transformation (55) for $m=3$ and F the $GF[p^n]$. The hypothesis is that its characteristic equation has no root in F. Changing the notation and applying §1, we may write it $V\,T_{3,\lambda}$, where V is a special Abelian transformation on ξ_1, η_1, ξ_2, η_2, and where, as always in this section, $\lambda^{p^n+1}=1$, $\lambda^2\neq 1$. Likewise for κ and τ.

Type $T_{1,\sigma}T_{2,\sigma^{p^n}}T_{3,\lambda}$, $\sigma^{p^{2n}+1}=1$, $\sigma^2\neq 1$. Each of the

$$\tfrac{1}{8}(p^{2n}-1)(p^n-1) \text{ or } \tfrac{1}{8}\,2^{3n}$$

non-conjugate forms is commutative with $(p^{2n}+1)(p^n+1)$ transformations.

Type $T_{1,\sigma}T_{2,\sigma^{p^n}}T_{3,\lambda}$, $\sigma^{p^{2n}-1}=1$, $\sigma^{p^n\pm 1}\neq 1$. Each of the

$$\tfrac{1}{8}(p^n-1)^3 \text{ or } \tfrac{1}{8}\,2^{2n}(2^n-2)$$

non-conjugate forms is commutative with $(p^{2n}-1)(p^n+1)$.

Type $T_{1,\kappa}T_{2,\tau}T_{3,\lambda}$, κ, κ^{-1}, τ, τ^{-1}, λ, λ^{-1} all distinct roots of $x^{p^n+1}=1$. The number of non-conjugate forms is

$$\tfrac{1}{48}(p^n-1)(p^n-3)(p^n-5) \text{ or } \tfrac{1}{48}\,2^n(2^n-2)(2^n-4).$$

Each is commutative with $(p^n+1)^3$ transformations.

Type $T_{1,\kappa}T_{2,\kappa}T_{3,\lambda}$, κ, κ^{-1}, λ, λ^{-1} all distinct. There are

$$\tfrac{1}{4}(p^n-1)(p^n-3) \text{ or } \tfrac{1}{4}\,2^n(2^n-2)$$

non-conjugate forms, each commutative with (Tr., §21)

$$(p^n+1)\,p^n(p^{2n}-1)\,.\,(p^n+1).$$

Type $T_{1,\lambda}T_{2,\lambda}T_{3,\lambda}$. There are $\frac{1}{2}(p^n-1)$ or $\frac{1}{2}\,2^n$ non-conjugate forms. By the general theory each is commutative with only S:

$$S=\begin{pmatrix} a & 0 & b & 0 & c & 0 \\ 0 & a^{p^n} & 0 & b^{p^n} & 0 & c^{p^n} \\ e & 0 & f & 0 & g & 0 \\ 0 & e^{p^n} & 0 & f^{p^n} & 0 & g^{p^n} \\ h & 0 & j & 0 & l & 0 \\ 0 & h^{p^n} & 0 & j^{p^n} & 0 & l^{p^n} \end{pmatrix},\quad \Sigma=\begin{pmatrix} a & b & c \\ e & f & g \\ h & j & l \end{pmatrix}.$$

The 6 conditions that S shall be Abelian are precisely the conditions that Σ shall be hyperorthogonal (Linear Groups, p. 133), the number of which is

$$(p^{3n}+1)p^{2n}(p^{2n}-1)p^{n}(p^{n}+1).$$

Type (63) $T_{3,\lambda}$, $\lambda \neq \kappa, \kappa^{-1}$. The number of non-conjugate forms is

$$\tfrac{1}{4}(p^{n}-1)(p^{n}-3) \text{ or } \tfrac{1}{4}2^{n}(2^{n}-2).$$

Each is commutative with $p^{n}(p^{n}+1)^{2}$ transformations.

Type (63) $T_{3,\kappa}$. There are $\tfrac{1}{2}(p^{n}-1)$ or $\tfrac{1}{2}2^{n}$ non-conjugate forms. Each is commutative with only (118), where now

$$a_1=a^{p^n},\quad b_1=b^{p^n},\quad c_1=c^{p^n},\quad d_1=d^{p^n},\quad e_1=e^{p^n}.$$

The resulting transformations are, however, not Abelian, the present case being unique in this respect. Indeed, by §19, end, a transformation of $SA(4, p^{n})$ is reducible to its canonical form (63) by a special Abelian transformation in the $GF[p^{2n}]$. It is otherwise with $T_{3,\kappa}$, $\kappa^{p^n+1}=1$. It follows from §1 that we may assume as the transformation of variables to obtain $T_{3,\kappa}$

$$x_3=(\xi_3-\gamma\kappa\eta_3)/f(\kappa),\quad y_3=(\xi_3-\gamma\kappa^{-1}\eta_3)/f(\kappa^{-1}),$$

$f(\kappa)$ being linear in κ with coefficients in the $GF[p^{n}]$. We can make

$$f(\kappa)\,.\,f(\kappa^{-1})=\gamma(\kappa-\kappa^{-1})^{2},$$

by choice of the coefficients in $f(\kappa)$. For our transformation of variables,

$$\sum_{i=1}^{3}\begin{vmatrix}\xi_i & \eta_i\\ \xi_i^{*} & \eta_i^{*}\end{vmatrix}=\begin{vmatrix}x_1 & y_1\\ x_1^{*} & y_1^{*}\end{vmatrix}+\begin{vmatrix}x_2 & y_2\\ x_2^{*} & y_2^{*}\end{vmatrix}+(\kappa-\kappa^{-1})\begin{vmatrix}x_3 & y_3\\ x_3^{*} & y_3^{*}\end{vmatrix}.$$

Transformation (118) must leave the second member unaltered. Hence*

$$a^{p^n+1}=1,\quad e^{p^n+1}=1,\quad ab^{p^n}-a^{p^n}b+(\kappa-\kappa^{-1})\,d^{p^n+1}=0,$$
$$ac^{p^n}+(\kappa-\kappa^{-1})\,de^{p^n}=0,\quad a^{p^n}c-(\kappa-\kappa^{-1})\,d^{p^n}e=0.$$

One of the last two may be dropped. The other determines c. Now

$$d^{p^n+1}=\left[\left(\frac{b}{a}\right)^{p^n}-\frac{b}{a}\right]\Bigg|(\kappa^{-1}-\kappa).$$

For any b/a in the $GF[p^{2n}]$, the second member equals its conjugate, and hence belongs to the $GF[p^{n}]$. Hence if b/a is not in the $GF[p^{n}]$, there are exactly $p^{n}+1$ roots d in the $GF[p^{2n}]$; otherwise, $d=0$. The number of transformations of $SA(6, p^n)$ commutative with one having the canonical form (63) $T_{3,\kappa}$ is therefore *

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* Modifying §2 for $m=3$, we define C'_{rs} from C_{rs} by inserting the factor $\kappa-\kappa^{-1}$ before the term given by $l=3$ in (3). Then $C'_{12}=C'_{34}=1$, $C'_{56}=\kappa-\kappa^{-1}$; all remaining $C'_{rs}=0$.

$$(p^n+1)^2\,[p^n+(p^{2n}-p^n)(p^n+1)]\equiv p^{3n}\,(p^n+1)^2.$$

The remaining types in §§7–33 for $m=3$.

41. We now discuss for $m=3$ the canonical forms determined in §§7–33, aside from types (16), (20), (23) and (55), which have been exhaustively treated in §§34-40.

Type (22). There are $\frac{1}{2}(p^n-3)$ or $\frac{1}{2}(2^n-2)$ non-conjugate forms. Indeed, $P_{12}M_3^{-1}T_{2,-1}Q_{3,1,1}$ transforms (22) into a transformation derived from (22) by replacing κ by κ^{-1} throughout. The only linear transformations (13) commutative with (22) are

$$\begin{pmatrix} \alpha_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_{11} & \beta_{12} & 0 & 0 & \delta_{13} \\ 0 & 0 & \delta_{11} & 0 & 0 & 0 \\ \beta_{21} & 0 & 0 & \alpha_{11} & \beta_{23} & 0 \\ -\beta_{23} & 0 & 0 & 0 & \alpha_{11} & 0 \\ 0 & 0 & \delta_{13} & 0 & 0 & \delta_{11} \end{pmatrix}.$$

The Abelian conditions reduce to

$$\delta_{11}=\alpha_{11}^{-1},\quad \beta_{21}=\alpha_{11}^2\beta_{12}-\alpha_{11}^3\delta_{13}^2,\quad \beta_{23}=\alpha_{11}^2\delta_{13}.$$

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* I give a new proof for $p^n=2^n$, and a check for $p>2$. In each case *the work is carried out in the initial field.*

Let first p^n be of the form $4l+3$, so that -1 is a not-square. Then the special Abelian transformation $\Sigma\equiv P_{12}L_{1,-1}T_{1,-1}M_3$ has the canonical form (63) $T_{3,\kappa}$, $\kappa^2=-1$. If (13) be commutative with Σ, it must be commutative with $\Sigma^4=L_{1,1}L_{2,1}$ and hence have the form of the second matrix just preceding (123), for $\mu=1$. The latter is commutative with Σ if, and only if,

$$\alpha_{21}=-\delta_{12},\quad \alpha_{22}=\alpha_{11},\quad \gamma_{22}=\gamma_{11},\quad \gamma_{32}=\delta_{31},\quad \delta_{32}=-\gamma_{31},$$
$$\alpha_{23}=\gamma_{13},\quad \gamma_{23}=-\alpha_{13},\quad \beta_{33}=-\gamma_{33},\quad \delta_{33}=\alpha_{33},\quad \gamma_{12}=\alpha_{21}-\gamma_{21}.$$

The Abelian conditions then reduce to

$$\alpha_{11}^2+\delta_{12}^2=1,\quad \alpha_{33}^2+\gamma_{33}^2=1,\quad \alpha_{11}\delta_{12}+2\alpha_{11}\gamma_{12}-2\gamma_{11}\delta_{12}+\alpha_{13}^2+\gamma_{13}^2=0,$$
$$\alpha_{11}\gamma_{31}+\delta_{12}\delta_{31}+\alpha_{13}\gamma_{33}-\gamma_{13}\alpha_{33}=0,\quad \alpha_{11}\delta_{31}-\delta_{12}\gamma_{31}+\alpha_{13}\alpha_{33}+\gamma_{13}\gamma_{33}=0.$$

The last two determine α_{13} and γ_{13}, the determinant of their coefficients being unity. Since $p>2$, the third determines either γ_{12} or γ_{11}, which do not occur in the other conditions. Now $x^2+y^2=1$ has p^n+1 sets of solutions in the field of order $p^n=4l+3$. Since γ_{31}, δ_{31} and either γ_{12} or γ_{11} remain arbitrary, there are $p^{3n}(p^n+1)^2$ sets of solutions.

For the $GF[2^n]$, a transformation of $SA(6, 2^n)$ having the canonical form (63) $T_{3,\kappa}$ is

$$\xi_1'=\eta_1,\quad \eta_1'=\xi_1+\delta\eta_1+\xi_2,\quad \xi_2'=\eta+\delta\xi_2+\eta_2,\quad \eta_2'=\xi_2,\quad \xi_3'=\eta\ ,\quad \eta_3'=\xi_3+(\delta+1)\,\eta_3,$$

The number of transformations commutative with (22) is thus $p^{2n}(p^n - 1)$.

Type $R_\beta T_{3,\lambda}$, $p = 2$, where (as below) $\lambda^{p^n+1} = 1$, $\lambda^2 \neq 1$. The number of non-conjugate forms is 2^n. Each is commutative (Trans., p. 112) with only $2 \cdot 2^{2n}(2^n + 1)$ Abelian transformations.

Type $R_{1,2,1}T_{3,\lambda}$, $p = 2$. There are $\frac{1}{2}2^n$ non-conjugate forms, each commutative with $2^{4n}(2^{2n} - 1)(2^n + 1)$ (Trans., p. 111).

Type $A_\mu T_{1,\pm 1}T_{2,\pm 1}T_{3,\lambda}$, $p > 2$. There are $2(p^n - 1)$ non-conjugate forms, each commutative with $2p^{2n}(p^n + 1)$ (Tr., p. 113).

Type $S_{0,1,-1}T_{1,\pm 1}T_{2,\pm 1}T_{3,\pm 1}$, $p > 2$. An Abelian transformation commutative with either of the two is of the form (36), subject to the conditions (37), (38), R_{12}, R_{24}, R_{34}, R_{23} of §11, where now $b = c = 0$ (it being allowable to substitute R_{23} for R_{46}). Let

$$\beta_{22} + \beta_{23} + \beta_{32} + \beta_{33} = B, \quad \beta_{22} + \beta_{23} = \rho, \quad \beta_{23} + \beta_{33} = \sigma, \quad \beta_{12} + \beta_{13} = \varkappa. \qquad (136)$$

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whose characteristic determinant is the cube of $\rho^2 - (\delta + 1)\rho + 1$, which is assumed to be irreducible in the field. Forming the conditions that (13) $S = S(13)$, we find that (13) must have the form

$$\begin{pmatrix} a_{11} & \gamma_{11} & a_{12} & \gamma_{12} & a_{13} & \gamma_{13} \\ \gamma_{11} & a_{11} + \delta\gamma_{11} + a_{12} & \gamma_{11} + \delta a_{12} + \gamma_{12} & a_{12} & \gamma_{13} & a_{13} + (\delta + 1)\gamma_{13} \\ a_{12} & \gamma_{11} + \delta a_{12} + \gamma_{12} & a_{11} + \delta\gamma_{11} + a_{12} & \gamma_{11} & \gamma_{13} & a_{13} + (\delta + 1)\gamma_{13} \\ \gamma_{12} & a_{12} & \gamma_{11} & a_{11} & a_{13} & \gamma_{13} \\ a_{31} & \gamma_{31} & \gamma_{31} & a_{31} & a_{33} & \gamma_{33} \\ \gamma_{31} & a_{31} + (\delta + 1)\gamma_{31} & a_{31} + (\delta + 1)\gamma_{31} & \gamma_{31} & \gamma_{33} & a_{33} + (\delta + 1)\gamma_{33} \end{pmatrix}.$$

Consider the Abelian conditions on the latter. We have

$$R_{56} \equiv a_{33}^2 + (\delta + 1)a_{33}\gamma_{33} + \gamma_{33}^2 = 1, \quad R_{14} \equiv 0, \quad R_{23} \equiv 0,$$

$$R_{24} \equiv R_{13} \equiv a_{11}\gamma_{11} + \delta a_{11}a_{12} + \delta\gamma_{11}\gamma_{12} + a_{12}\gamma_{12} + a_{13}^2 + (\delta + 1)a_{13}\gamma_{13} + \gamma_{13}^2 = 0,$$

$$(s) \quad R_{12} + R_{13} \equiv (a_{11} + \gamma_{12})^2 + (\delta + 1)(a_{11} + \gamma_{12})(\gamma_{11} + a_{12}) + (\gamma_{11} + a_{12})^2 = 1,$$

$$R_{16} \equiv a_{11}a_{31} + \gamma_{11}\gamma_{31} + a_{12}\gamma_{31} + \gamma_{12}a_{31} + a_{13}a_{33} + \gamma_{13}\gamma_{33} + (\delta + 1)(a_{11}\gamma_{31} + \gamma_{12}\gamma_{31} + a_{13}\gamma_{33}) = 0,$$

$$R_{15} \equiv a_{11}\gamma_{31} + a_{12}a_{31} + a_{13}\gamma_{33} + \gamma_{11}a_{31} + \gamma_{12}\gamma_{31} + \gamma_{13}a_{33} = 0,$$

$$R_{34} \equiv R_{12}, \; R_{36} \equiv R_{26} \equiv R_{45} \equiv R_{15}, \quad R_{46} \equiv R_{16},$$

$$R_{16} + R_{25} \equiv (\delta + 1)R_{15}, \quad R_{35} \equiv R_{25}.$$

Hence all follow from R_{56}, R_{13}, (s), R_{16}, R_{15}. If in R_{16} and R_{15} we collect the terms in γ_{31}, a_{31}, we find that the determinant of their coefficients is unity by (s). Hence R_{16} and R_{15} serve to determine γ_{31}, a_{31}, which occur in neither R_{56} nor R_{13}. Now R_{56} has $2^n + 1$ sets of solutions a_{33}, γ_{33}; (s) has $2^n + 1$ sets of solutions $\lambda \equiv a_{11} + \gamma_{12}$, $\mu \equiv \gamma_{11} + a_{12}$, not both zero. Then R_{13} becomes

$$\mu a_{11}(\delta + 1) + \lambda\gamma_{11}(\delta + 1) + \lambda\mu + a_{13}^2 + (\delta + 1)a_{13}\gamma_{13} + \gamma_{13}^2 = 0,$$

and hence determines a_{11} or else γ_{11}. But a_{13}, γ_{13} and a_{11} or else γ_{11} remain arbitrary. Hence there are $2^{3n}(2^n + 1)$ transformations.

Then R_{12}, (37), (38), R_{34}, R_{23} become, respectively,

$$\alpha_{11}\delta_{11} - \gamma B = 1, \quad \beta_{21} + \beta_{31} = B - x, \quad \delta_{11}(2x - B) + B(2\sigma - B) = 0, \qquad (137)$$

$$2\alpha\alpha_{11} - 2\gamma\rho = 1 + \alpha_{11}\gamma + \alpha_{11}^2, \quad -\alpha B + \delta_{11}\rho = -\alpha_{11}\sigma - \gamma x. \qquad (138)$$

The determinant of the coefficients of α and ρ in the last two equals 2 by $(137)_1$. Hence they determine α, ρ in terms of α_{11}, δ_{11}, γ, B, σ, x.

Let first δ_{11} be given any one of the $p^n - 1$ marks $\neq 0$, while γ, B, σ, β_{11}, β_{12}, β_{33} are given any one of the p^{6n} sets of 6 marks. Then α_{11} is determined by $(137)_1$, and x by $(137)_3$. Then, as above, α and ρ are determined by (138). Then conditions (136), read in reverse order, determine β_{13}, β_{23}, β_{22}, β_{32}. Then R_{24} determines β_{21}, and $(137)_2$ determines β_{31}. All the conditions have been satisfied.

Let next $\delta_{11} = 0$, and give α_{11}, x, β_{13}, β_{31}, β_{33} any values, γ any value $\neq 0$. Then $(137)_1$ gives $B = -\gamma^{-1}$, $(137)_3$ gives $\sigma = -\frac{1}{2}\gamma^{-1}$. Then, as above, (138) determine α and ρ. Next, $(137)_2$ determines β_{21}; while (136), read in reverse order, determine β_{12}, β_{23}, β_{22}, β_{32}. Then R_{24} determines β_{11}.

For $p > 2$, *the number of Abelian transformations commutative with* $S_{0,1,-1}$ *is*

$$(p^n - 1)\,p^{6n} + (p^n - 1)\,p^{5n} = p^{5n}(p^{2n} - 1).$$

Type $S_{b,1,1}$, $p = 2$, $b = 0$ or a particular root of $f(\eta) = 1$, f defined by (28). Every Abelian transformation commutative with $S_{b,1,1}$ is given by (36), subject to (39), (40) and R_{24}, where now $c = b$.

Let first $b = 0$, so that $B = 0$ or δ_{11}. For $B = 0$,

$$\alpha_{11}\delta_{11} = 1, \quad \gamma = \alpha_{11} + \delta_{11}, \quad \beta_{12} + \beta_{21} + \beta_{13} + \beta_{31} = 0,$$
$$\gamma(\beta_{21} + \beta_{31} + \beta_{23}) + \delta_{11}\beta_{22} + \alpha_{11}\beta_{33} = 0,$$

together with R_{24}. If $\gamma = 0$, then $\alpha_{11} = \delta_{11} = 1$, $\beta_{22} = \beta_{33}$, $\beta_{32} = \beta_{23}$; we may take β_{33}, β_{23}, β_{11}, β_{12}, β_{13}, α arbitray, when β_{21} is determined by R_{24}. If $\gamma \neq 0$, we may take β_{33}, β_{22}, β_{21}, β_{12}, β_{13}, α arbitrary, α_{11} any mark $\neq 0$, 1, whence $\alpha_{11} = 1/\delta_{11}$, $\beta_{31} = \beta_{12} + \beta_{21} + \beta_{13}$, while β_{23} is determined by the equation above, β_{32} by $B = 0$, β_{11} by R_{24}. Hence the case $B = 0$ yields $2^{6n} + 2^{6n}(2^n - 2)$ sets For $B = \delta_{11}$, (39) give

$$\delta_{11} = \alpha_{11} \neq 0, \quad \gamma = \alpha_{11}^{-1} + \alpha_{11}, \quad \beta_{12} + \beta_{13} + \beta_{21} + \beta_{31} = \alpha_{11}.$$

We may take β_{11}, β_{12}, β_{13}, β_{23}, β_{32}, β_{33} arbitrary, α_{11} any mark $\neq 0$. Then β_{22} is determined by $B = \delta_{11}$, β_{21} by R_{24}, β_{31} by the third equation above, α by $(40)_1$.

Combining these $(2^n - 1)\,2^{6n}$ sets with those for $B = 0$, we get* $2\,(2^n - 1)\,2^{6n}$ *as the total number for* $b = 0$.

Let next $f(b) = 1$, so that $b \neq 0$. If $\delta_{11} = 0$,

$$B^2 = b, \quad \gamma = B^{-1}, \quad \alpha_{11} = 0 \text{ or } \gamma.$$

We may take $\beta_{12}, \beta_{21}, \beta_{13}, \beta_{23}, \beta_{32}, \beta_{33}$ arbitrary. Then $(39)_1$ determines β_{31}, $B = b^{\frac{1}{2}}$ determines β_{22}, $(40)_1$ determines α, R_{24} determines β_{11}. There result $2 \,.\, 2^{6n}$ sets. If $\delta_{11} \neq 0$, $(40)_2$ gives

$$(B\delta_{11}^{-1})^2 + B\delta_{11}^{-1} = b + b\delta_{11}^{-2}.$$

Hence, by §9, must $f(b + b\delta_{11}^{-2}) = 0$, whence $f(b\delta_{11}^{-2}) = 1$. The latter has 2^{n-1} roots δ_{11} in the $GF[2^n]$. To each of these correspond two roots B in the field. Eliminating α_{11} between the last two equations (39) and simplifying by $(40)_2$, we get

$$b\gamma^2 + \gamma\delta_{11} = 1 + \delta_{11}^2.$$

Combining this with $(40)_2$, we get $b\gamma = B$ or $B + \delta_{11}$. Hence there are precisely 2^{n+1} sets α_{11}, $\delta_{11} \neq 0$, B, γ satisfying $(39)_1$, $(39)_2$, $(40)_2$. Of these exactly two sets have $\gamma = 0$:

$$\alpha_{11} = \delta_{11} = 1, \quad B = 0, \quad \gamma = 0; \quad \alpha_{11} = \delta_{11} = 1, \quad B = 1, \quad \gamma = 0.$$

For $\gamma = 0$, we may take $\alpha, \beta_{23}, \beta_{33}, \beta_{11}, \beta_{12}, \beta_{13}$ arbitrary, when $\beta_{22}, \beta_{32}, \beta_{21}, \beta_{31}$ are determined by $(40)_1$, B, R_{24}, $(39)_1$, respectively. For $\gamma \neq 0$, we may take $\alpha, \beta_{23}, \beta_{33}, \beta_{32}, \beta_{31}, \beta_{13}$ arbitrary, when $\beta_{22}, \beta_{21}, \beta_{12}\ \beta_{11}$ are determined by B, $(40)_1$, $(39)_1$, R_{24}, respectively. There result $2^{n+1} \,.\, 2^{6n}$ sets for $\delta_{11} \neq 0$. *Hence,*† *there are* $2\,(2^n + 1)\,2^{6n}$ *sets for* $f(b) = 1$.

Type $S_{0,\,\mu}T_{1,\,\pm 1}T_{2,\,\pm 1}T_{3,\,\pm 1}$, $p > 2$. An Abelian transformation commutative with one of the four forms must have the form (44), subject to (45)–(48), where now $B = \beta' = 0$, $A = \alpha = \mu$. Hence

$$\alpha_{11} = \delta_{11} = \pm 1, \quad \alpha_{21} = -\delta_{12}, \quad \alpha_{31} = \delta_{12} + \delta_{13}, \quad \beta_{31} = \beta_{13} + 2\mu\delta_{13} + \mu\delta_{12},$$
$$\beta_{21} = \mu\delta_{13} - \beta_{12} + \beta_{13}^2, \quad 2\delta_{13} = \pm\,\delta_{12}^2 - \delta_{12},$$

together with (48), which determines β_{12}. Since $\beta_{11}, \delta_{12}, \beta_{13}$ remain arbitrary, there are $2p^{3n}$ transformations.

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* I note for check that I had obtained this number as the number of Abelian transformations commutative with $V_{0,1,0}$ which (§15) is conjugate with $S_{0,1,1}$.

† Obtained also by a different but longer method of solution.

Type $W_{0,\beta}$, $p=2$, $\beta=0$ or a particular root of $f(\eta)=1$. An Abelian transformation commutative with $W_{0,\beta}$ must have the form (50), subject to (51) and (52), where $D=\delta=0$, $B=\beta$. The $2\,.\,2^{3n}$ sets are given as follows

$$\alpha_{11}=0,\quad \alpha_{12}=1,\quad \delta_{13}=\beta_{13}+\beta_{13}^{\frac{1}{2}},\quad \beta_{11},\,\beta_{12},\,\beta_{13}\text{ arbitrary};$$
$$\alpha_{11}=1,\quad \alpha_{12}=0,\quad \beta_{13}=\delta_{13}+\delta_{13}^{\frac{1}{2}},\quad \beta_{11},\,\beta_{12},\,\delta_{13}\text{ arbitrary}.$$

Type $T_{1,\kappa}T_{2,\kappa^{p^n}}T_{3,\kappa^{p^{2n}}}$, $\kappa^{p^{3n}-1}=1$, $\kappa^{p^n-1}\neq 1$, furnished by (90). The six multipliers are all distinct* by §25. The replacement of κ by κ^{p^n} is effected by transforming by $(\xi_1\xi_2\xi_3)(\eta_1\eta_2\eta_3)$; the replacement of κ by κ^{-1}, by $M_1M_2M_3$. Hence there are $\frac{1}{6}\,(p^{3n}-p^n)$ non-conjugate forms. Each is commutative with $p^{3n}-1$ Abelian transformations having simultaneously the canonical form $T_{1,a}T_{2,a^{p^n}}$, $T_{3,a^{p^{2n}}}$, $a^{p^{3n}-1}=1$.

Type $T_{1,\kappa}T_{2,\kappa^{p^n}}T_{3,\kappa^{p^{2n}}}$, $\kappa^{p^{6n}}=\kappa$, $\kappa^{p^{3n}}\neq\kappa$, $\kappa^{p^{2n}}\neq\kappa$, furnished by (91). Now $\kappa^{-1}=\kappa^{p^{3n}}$ by §26, end. Of the $p^{3n}+1$ solutions, it is only necessary to exclude those making $\kappa^{p^n+1}=1$. Hence, as for preceding type, there are $\frac{1}{6}\,(p^{3n}-p^n)$ non-conjugate forms. Each is commutative with $p^{3n}+1$.

Type (100) for $\sigma=1/(\kappa^{-1}-\kappa)$, $\kappa^{p^n+1}=1$, $\kappa^2\neq 1$. Now $M_1P_{23}M_2M_3=P$ transforms (100) into a transformation which may be derived directly from (100) by replacing κ by κ^{-1}, σ by $\bar{\sigma}$, throughout. Now the transformer P does not preserve the conjugacy of variables since, for it, $\bar{x}_i=-y_i$. Now the transformation $x_i'=x_i$, $y_i'=-y_i\,(i=1,2,3)$ multiplies the bilinear function $\Sigma\,(x_iy_i^*-y_ix_i^*)$ by -1. But by the interchange of κ and κ^{-1}, the sign of the second member (101) is altered. Hence if we return from the canonical variables x_i, y_i to the original variables ξ_i, η_i, $(100)_\kappa$ and $(100)_{\kappa^{-1}}$ become transformations conjugate within the special Abelian group. Hence there are exactly $\frac{1}{2}\,(p^n-1)$ or $\frac{1}{2}\,2^n$ non-conjugate forms of the present type. If we determine the general ternary tarnsformation S commutative with the transformation which (100) effects on x_1, x_2, x_3, and take the conjugate transformation $\bar{S}$ on y_1, y_3, y_2, we obtain as the most general transformation commutative with (100)

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* To prove directly, if $\kappa^{-1}=\kappa^{p^n}$, then $\kappa^{p^{2n}}=\kappa^{-p^n}=\kappa$, whence $\kappa=\kappa^{p^n}$. If $\kappa^{-1}=\kappa^{p^{2n}}$, then $\kappa=\kappa^{p^{3n}}=\kappa^{-p^n}$, contrary to the preceding.

$$\begin{pmatrix} a & 0 & b & 0 & 0 & 0 \\ 0 & a^{p^n} & 0 & 0 & 0 & b^{p^n} \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & -b^{p^n} & 0 & a^{p^n} & 0 & c^{p^n} \\ -b & 0 & c & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & a^{p^n} \end{pmatrix}.$$

The Abelian conditions on it reduce to the three:

$$aa^{p^n} = 1, \quad -ab^{p^n} + ba^{p^n} = 0, \quad -bb^{p^n} - a^{p^n}c - ac^{p^n} = 0.$$

Hence b/a belongs to the $GF[p^{2n}]$. The third condition may be written

$$\left(\frac{c}{a}\right)^{p^n} + \frac{c}{a} = b^{p^n+1}.$$

Subtracting it from the power of p^n it, we see that it has p^n solutions c/a in the $GF[p^{2n}]$, when b belongs to that field. Hence* there are exactly $p^{2n}(p^n+1)$ transformations in G which are commutative with one having the canonical form (100).

Checks. Table for $p^n = 2$, $m = 3$.

42. The complete list of canonical types for $m = 3$ is given in §§37–41, together with the number N of non-conjugate canonical forms of each type and the number C of transformations of the group G which are commutative with each form. The order of G for the $GF[p^n]$ is

$$\Omega \equiv p^{9n}(p^{6n} - 1)(p^{4n} - 1)(p^{2n} - 1).$$

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* As a check, I verified that the only Abelian substitutions in the GF [2] commutative with (75) for $a = \delta = 1$, and necessarily $\gamma = g = 1$, are the 12:

$$\begin{pmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & 0 & \alpha_{13} & 0 \\ \gamma_{11} & \alpha_{11} + \gamma_{11} & \alpha_{12} + \alpha_{13} + \gamma_{11} & 0 & \alpha_{12} & 0 \\ 0 & 0 & \alpha_{11} + \gamma_{11} & 0 & \gamma_{11} & 0 \\ \alpha_{13} & \alpha_{12} & \beta_{22} & \alpha_{11} & \beta_{23} & \gamma_{11} \\ 0 & 0 & \gamma_{11} & 0 & \alpha_{11} & 0 \\ \alpha_{12} & \alpha_{12} + \alpha_{13} + \gamma_{11} & \alpha_{12} + \beta_{22} + \beta_{23} & \gamma_{11} & \beta_{22} & \alpha_{11} + \gamma_{11} \end{pmatrix},$$

where, for $\gamma_{11} = 0$, we get $\alpha_{11} = 1$, $\alpha_{12} = 0$, $\beta_{22} = \alpha_{13}$, β_{23} being arbitrary; for $\gamma_{11} = 1$,

$$\alpha_{13} = \alpha_{11} + \alpha_{11}\alpha_{12}, \quad \beta_{23} = \alpha_{11}\alpha_{12} + \alpha_{11}\beta_{22} + \alpha_{12}.$$

Further, I verified that (75) is now of period 12.

One check on the work is that C is found to divide Ω for each type. A more fundamental check is furnished by equality of Ω and the sum of the expressions $N\Omega \div C$ (the total number of transformations reducible to a given *type*) for the complete list of types. This lengthy computation was made for $p > 2$ with the greatest care, each step being checked.* The resulting polynomial of degree 21 in p^n was found to be identical with Ω, thus furnishing a 21-fold check. This check was particularly convincing as the individual expressions to be added involved fractions which, in their "lowest terms," had as denominators the various factors of 48.

The case $p = 2$ was checked only when $n = 1$ (see accompanying table, the total in the last column being $1451520 = 2^9.3^4.5.7 = \Omega$). In fact, the separation of the cases $p = 2, p > 2$ is of trivial character except for a few types.

Independent of the preceding checks, I convinced myself that no two distinct canonical types in the list represented transformations conjugate within G.

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* The expressions were left in factored forms so that many sets could be readily added. Those in §§39-41 were twice added with the same result.

Table of non-conjugate types of operators of $SA(6, 2)$.

Type	Period	Commutative	Conjugate
Identity	1	all	1
$L_{11}L_{21}L_{31}$	2	$2^7.3$	3780
L_{11}	2	$2^9.3^2.5$	63
$L_{11}L_{21}$	2	$2^9.3$	945
R_{121}	2	$2^9.3^2$	315
$R_0L'_{31}$	4	2^5	45360
R_0	4	$2^6.3$	7560
R_1	4	$2^6.3$	7560
$T_{2,\lambda}(\lambda^2=\lambda+1)$	3	$2^4.3^3.5$	672
$L_{11}T_{2\lambda}$	6	$2^4.3^2$	10080
$L_{11}T_{2\lambda}L_{31}$	6	$2^4.3$	30240
$T_{1\sigma}T_{2\sigma^2}(\sigma^5=1)$	5	$2.3.5$	48384
$T_{1\sigma}T_{2\sigma^2}L_{31}$	10	2.5	145152
$T_{1\kappa}T_{2\kappa}(\varkappa^2=\varkappa+1)$	3	$2^2.3^3$	13440
$T_{1\kappa}T_{2\kappa}L_{31}$	6	$2^2.3^2$	40320
$(63)\ L_{11}$	6	$2^2.3$	120960
(63)	6	$2^2.3^2$	40320
$T_{1\sigma}T_{2\sigma^2}T_{3\lambda}$	15	3.5	96768
$T_{1\lambda}T_{2\lambda}T_{3\lambda}$	3	$2^3.3^4$	2240
$(63)\ T_{3\kappa}$	6	$2^3.3^2$	20160
$R_0T_{3\lambda}$	12	$2^3.3$	60480
$R_1T_{3\lambda}$	12	$2^3.3$	60480
$R_{121}T_{3\lambda}$	6	$2^4.3^2$	10080
S_{111}	4	$2^7.3$	3780
S_{011}	4	2^7	11340
W_{00}	8	2^4	90720
W_{01}	8	2^4	90720
$T_{1\rho}T_{2\rho^2}T_{3\rho^4}(\rho^7=1)$	7	7	207360
$T_{1\rho}T_{2\rho^2}T_{3\rho^4}(\rho^9=1)$	9	3^2	161280
$(100), (\varkappa^3=1, \sigma=1)$	12	$2^2.3$	120960

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